IFF

A formula must be found for the n^{th} triangle number. Let the function $f: \mathbb{N} \to T$, where T is the set of all triangle numbers exist such that $f: n \mapsto T_n$, where T_n is the n^{th} triangle number. Then

$$f: 1 \mapsto 1$$
$$f: 2 \mapsto 3 = 1 + 2 = 1 + (1 + 1)$$
$$f: 3 \mapsto 6 = 1 + 2 + 3 = 1 + \big(1 + 1(1)\big) + \big(1 + 2(1)\big)$$
$$f: n \mapsto \sum_{i=1}^{n} i$$

We see that $\sum_{i=1}^{n} i$ forms an arithmetic sequence, with the first term being 1, the last term being n and the number of terms being n. Therefore

$$f: n \mapsto \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

From this, we see that

$$8[f(n)] + 1$$
$$= \frac{8n(n+1)}{2} + 1$$
$$= 4n(n+1) + 1$$
$$= 4n^2 + 4n + 1$$
$$= (2n+1)^2$$

To prove the second conjecture, if $8n + 1$ is a square number, then

$$8n + 1 = a^2$$

and after rearrangement

$$n = \frac{a^2 - 1}{8}$$

$8n + 1$ can be written as $2(4n) + 1$ and since 4 and n are natural numbers and the set of natural numbers is closed under multiplication, $2(4n) + 1 = 2M + 1$, $M \in \mathbb{N}$, so $8n + 1$ is odd. Therefore, a^2 is odd, and also a is odd, since only the product of two odd numbers yields an odd number. So $a = 2M + 1$, then

$$n = \frac{(2M+1)^2 - 1}{8}$$
$$n = \frac{4M^2 + 4M + 1 - 1}{8}$$
$$n = \frac{4M^2 + 4M}{8}$$
$$n = \frac{4M(M+1)}{8}$$
$$n = \frac{M(M+1)}{2}$$

Thus, $8n + 1$ is a square number iff n is a triangle number

Another way of phrasing the above is if $8n + 1$ is a square number, n is a triangle number.

We can use this fact to test if numbers are triangle numbers. Multiply the number by 8 and add 1 to the result. Square root this result. If the final result is a natural number, then the original number was a triangle number. Using this method, 3655 and 7626 are triangle numbers.