Farey Sequences Questions

So far, all the Farey Sequences except $F1$ have contained an odd number of fractions. Can you find a Farey Sequence with an even number of fractions?

Solution by Prithvi, Shrish and Hardik from Bangkok Patana School.

We have found out from a various number of trial and error experiments that all farey sequences excluding F_1 have an odd number of fractions.

Here are some examples:

$$F_n$$

We can take F_9 as an example (denominator upto a square number), the farey sequence for that is:

$0, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, 1$

In this there are 29 fractions.

We can also take F_{36}(denominator upto a number with a square number of factors). Its farey sequence is:

$0, \frac{1}{36} \frac{1}{35}, \frac{1}{34}, \frac{1}{33}, \frac{1}{32}, \frac{1}{31}, \frac{1}{30}, \frac{1}{29}, \frac{1}{28}, \frac{1}{27}, \frac{1}{26}, \frac{1}{25}, \frac{1}{24}, \frac{1}{23} \frac{1}{22}, \frac{1}{21}, \frac{1}{20}, \frac{1}{19}, \frac{1}{18}, \frac{2}{35}, \frac{1}{17}, \frac{2}{33}, \frac{1}{16}, \frac{2}{31}, \frac{1}{15}, \frac{2}{29}, \frac{1}{14},$
$\frac{2}{27}, \frac{1}{13}, \frac{2}{25}, \frac{1}{12}, \frac{3}{35}, \frac{2}{23}, \frac{3}{34}, \frac{1}{11}, \frac{3}{32}, \frac{2}{21}, \frac{3}{31}, \frac{1}{10}, \frac{3}{29}, \frac{2}{19}, \frac{3}{28}, \frac{1}{9}, \frac{4}{35}, \frac{3}{26}, \frac{2}{17}, \frac{3}{25}, \frac{4}{33}, \frac{1}{8}, \frac{4}{31}, \frac{3}{23}, \frac{2}{15}, \frac{3}{22}, \frac{4}{29}, \frac{5}{36}, \frac{1}{7}, \frac{5}{34},$

$\frac{4}{27}, \frac{3}{20}, \frac{5}{33}, \frac{2}{13}, \frac{5}{32}, \frac{3}{19}, \frac{4}{25}, \frac{5}{31}, \frac{1}{6}, \frac{6}{35}, \frac{5}{29}, \frac{4}{23}, \frac{3}{17}, \frac{5}{28}, \frac{2}{11}, \frac{5}{27}, \frac{3}{16}, \frac{4}{21}, \frac{5}{26}, \frac{6}{31}, \frac{7}{36}, \frac{1}{5}, \frac{7}{34}, \frac{6}{29}, \frac{5}{24}, \frac{4}{19}, \frac{7}{33}, \frac{3}{14}, \frac{5}{23}, \frac{7}{32},$
$\frac{2}{9}, \frac{7}{31}, \frac{5}{22}, \frac{8}{35}, \frac{3}{13}, \frac{7}{30}, \frac{4}{17}, \frac{5}{21}, \frac{6}{25}, \frac{7}{29}, \frac{8}{33}, \frac{1}{4}, \frac{9}{35}, \frac{8}{31}, \frac{7}{27}, \frac{6}{23}, \frac{5}{19}, \frac{9}{34}, \frac{4}{15}, \frac{7}{26}, \frac{3}{11}, \frac{8}{29}, \frac{5}{18}, \frac{7}{25}, \frac{9}{32}, \frac{2}{7}, \frac{9}{31}, \frac{7}{24}, \frac{5}{17},$
$\frac{8}{27}, \frac{3}{10}, \frac{10}{33}, \frac{7}{23}, \frac{11}{36}, \frac{4}{13}, \frac{9}{29}, \frac{5}{16}, \frac{11}{35}, \frac{6}{19}, \frac{7}{22}, \frac{8}{25}, \frac{9}{28}, \frac{10}{31}, \frac{11}{34}, \frac{1}{3}, \frac{12}{35}, \frac{11}{35}, \frac{10}{29}, \frac{9}{26}, \frac{8}{23}, \frac{7}{20}, \frac{6}{17}, \frac{11}{31}, \frac{5}{14}, \frac{9}{25}, \frac{13}{36}, \frac{4}{11}, \frac{11}{30},$
$\frac{7}{19}, \frac{10}{27}, \frac{13}{35}, \frac{3}{8}, \frac{11}{29}, \frac{8}{21}, \frac{13}{34}, \frac{5}{13}, \frac{12}{31}, \frac{7}{18}, \frac{9}{23}, \frac{11}{28}, \frac{13}{33}, \frac{2}{5}, \frac{13}{32}, \frac{11}{27}, \frac{9}{22}, \frac{7}{17}, \frac{12}{29}, \frac{5}{12}, \frac{12}{31}, \frac{8}{19}, \frac{11}{26}, \frac{14}{33}, \frac{3}{7}, \frac{13}{30}, \frac{10}{23}, \frac{7}{16}, \frac{11}{25},$
$\frac{15}{34}, \frac{4}{9}, \frac{13}{29}, \frac{9}{20}, \frac{14}{31}, \frac{5}{11}, \frac{16}{35}, \frac{11}{24}, \frac{6}{13}, \frac{13}{28}, \frac{7}{15}, \frac{15}{32}, \frac{8}{17}, \frac{17}{36}, \frac{9}{19}, \frac{10}{21}, \frac{11}{23}, \frac{12}{25}, \frac{13}{27}, \frac{14}{29}, \frac{15}{31}, \frac{16}{33}, \frac{17}{35}, \frac{1}{2} \frac{18}{35}, \frac{17}{33}, \frac{16}{31}, \frac{15}{29}, \frac{14}{27},$
$\frac{13}{25}, \frac{12}{23}, \frac{11}{21}, \frac{10}{19}, \frac{19}{36}, \frac{9}{17}, \frac{17}{32}, \frac{8}{15}, \frac{15}{28}, \frac{7}{13}, \frac{13}{24}, \frac{19}{35}, \frac{6}{11}, \frac{17}{31}, \frac{11}{20}, \frac{16}{29}, \frac{5}{9}, \frac{19}{34}, \frac{14}{25}, \frac{9}{16}, \frac{13}{23}, \frac{17}{30}, \frac{4}{7}, \frac{19}{33}, \frac{15}{26}, \frac{11}{19}, \frac{18}{31}, \frac{7}{12},$
$\frac{17}{29}, \frac{10}{17}, \frac{13}{22}, \frac{16}{27}, \frac{19}{32}, \frac{3}{5}, \frac{20}{33}, \frac{17}{28}, \frac{14}{23}, \frac{11}{18}, \frac{19}{31}, \frac{8}{13}, \frac{21}{34}, \frac{13}{21}, \frac{18}{29}, \frac{5}{8}, \frac{22}{35}, \frac{17}{27}, \frac{12}{19}, \frac{19}{30}, \frac{7}{11}, \frac{23}{36}, \frac{16}{25}, \frac{9}{14}, \frac{20}{31}, \frac{11}{17}, \frac{13}{20}, \frac{15}{23}, \frac{17}{26},$
$\frac{19}{29}, \frac{21}{32}, \frac{23}{35}, \frac{2}{3}, \frac{23}{34}, \frac{21}{31}, \frac{19}{28}, \frac{17}{25}, \frac{15}{22}, \frac{13}{19}, \frac{24}{35}, \frac{11}{16}, \frac{20}{29}, \frac{9}{13}, \frac{25}{36}, \frac{16}{23}, \frac{23}{33}, \frac{7}{10}, \frac{19}{27}, \frac{12}{17}, \frac{17}{24}, \frac{22}{31}, \frac{5}{7}, \frac{23}{32}, \frac{18}{25}, \frac{13}{18}, \frac{21}{29}, \frac{8}{11}, \frac{19}{26},$
$\frac{11}{15}, \frac{25}{34}, \frac{14}{19}, \frac{17}{23}, \frac{20}{27}, \frac{23}{31}, \frac{26}{35}, \frac{3}{4}, \frac{25}{33}, \frac{22}{29}, \frac{19}{25}, \frac{16}{21}, \frac{13}{17}, \frac{23}{30}, \frac{10}{13}, \frac{27}{35}, \frac{17}{22}, \frac{24}{31}, \frac{7}{8}, \frac{25}{32}, \frac{18}{23}, \frac{11}{14}, \frac{26}{33}, \frac{15}{19}, \frac{19}{24}, \frac{23}{29}, \frac{27}{34}, \frac{4}{5}, \frac{29}{36},$
$\frac{25}{31}, \frac{21}{26}, \frac{17}{21}, \frac{13}{16}, \frac{22}{27}, \frac{9}{11}, \frac{23}{28}, \frac{14}{17}, \frac{19}{23}, \frac{24}{29}, \frac{29}{35}, \frac{5}{6}, \frac{26}{31}, \frac{21}{25}, \frac{16}{19}, \frac{27}{32}, \frac{11}{13}, \frac{28}{33}, \frac{17}{20}, \frac{23}{27}, \frac{29}{34}, \frac{6}{7}, \frac{31}{36}, \frac{25}{29}, \frac{19}{22}, \frac{13}{15}, \frac{20}{23}, \frac{27}{31}, \frac{7}{8},$
$\frac{29}{33}, \frac{22}{25}, \frac{15}{17}, \frac{23}{26}, \frac{31}{35}, \frac{8}{9}, \frac{25}{28}, \frac{17}{19}, \frac{26}{29}, \frac{9}{10}, \frac{28}{31}, \frac{19}{21}, \frac{29}{32}, \frac{10}{11}, \frac{31}{34}, \frac{21}{23}, \frac{32}{35}, \frac{11}{12}, \frac{23}{25}, \frac{12}{13}, \frac{25}{27}, \frac{13}{14}, \frac{27}{29}, \frac{14}{15}, \frac{29}{31}, \frac{15}{16}, \frac{31}{33}, \frac{16}{17},$
$\frac{33}{35}, \frac{17}{18}, \frac{18}{19}, \frac{19}{20}, \frac{20}{21}, \frac{21}{22}, \frac{22}{23}, \frac{23}{24}, \frac{24}{25}, \frac{25}{26}, \frac{26}{27}, \frac{27}{28}, \frac{28}{29}, \frac{29}{30}, \frac{30}{31}, \frac{31}{32}, \frac{32}{33}, \frac{33}{34}, \frac{34}{35}, \frac{35}{36}, 1$

As you can see, there are 397 fractions in this too which means it is again an odd number

$F_8$is also another pivotal example as we wanted to try an even number. Its farey sequence is also listed below.

$0, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{5}{8}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, 1$

There are 23 fractions which is an odd number too.

To finish off our proof that farey sequences have an odd number of fractions, we usedF_7as 7 is a prime number and that was the last thing we wanted to try.

$0, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{2}{7}, \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{1}{2}, \frac{4}{7}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, 1$

In this, there is another odd number of fractions, 19.

However, we have made an interesting discovery. If in F_n, n= a prime number, then the farey sequence will have n-1 fractions added to it from the previous sequence. For example, F_6has 13 fractions, and F_7has 19 fraction which is an addition of (n-1) fractions since 13 + (7-1) = 19.