Nrich "In the Box" Solution

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Let r be the number of red , b the number of blue and t the total number of ribbons. To have a fair game we need:

$$P(rr) + P(bb) \; = \; P(rb) + P(br) \qquad \Rightarrow \qquad \frac{r}{t} \cdot \frac{r-1}{t-1} + \frac{b}{t} \cdot \frac{b-1}{t-1} \; = \; \frac{r}{t} \cdot \frac{b}{t-1} + \frac{b}{t} \cdot \frac{r}{t-1}$$

which easily simplifies to:

$$r(r-1) + b(b-1) \; = \; 2rb \tag{1}$$

We can choose to solve (1) as a quadratic with respect to r:

$$r^2 - (2b+1)r + b(b-1) = 0 \tag{2}$$

The determinant is:

$$D = (2b+1)^2 - 4b(b-1) \qquad \Rightarrow \qquad D = 8b+1$$

and the solutions will be:

$$r = \frac{2b+1 \pm \sqrt{8b+1}}{2}$$

Demanding integer solutions we see that the determinant must be a perfect square but since $2b+1$ is an odd number the determinant's square root must also be odd. Overall the determinant must be the square of an odd number, therefore we have:

$$D = (2n+1)^2 \qquad \Rightarrow \qquad 8b+1 = (2n+1)^2 \qquad n \in \mathbb{N}$$

which gives us the solutions for b and r:

$$\mathbf{b} = \frac{\mathbf{n(n+1)}}{\mathbf{2}} \qquad \textit{and} \qquad \mathbf{r} = \frac{\mathbf{n(n-1)}}{\mathbf{2}} \qquad \dagger$$

These are consecutive elements of the sequence of triangular numbers (0, 1, 3, 6, 10, 15, 21,...) which are produced by the following sum:

$$T_n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \binom{n+1}{2}$$

The last form is a *binomial coefficient*, it represents the number of distinct pairs that can be selected from $n+1$ objects, and it is read aloud as "n plus one choose two".

More information on triangular numbers can be found on wikipedia.

† The other solution for r simply gives $\frac{(n+1)(n+2)}{2}$ which is the next instead of the previous triangular number.