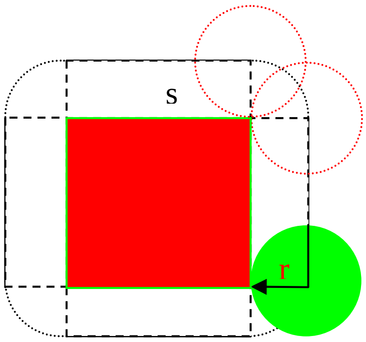
Solution to Roundabout

By Oliver Feng

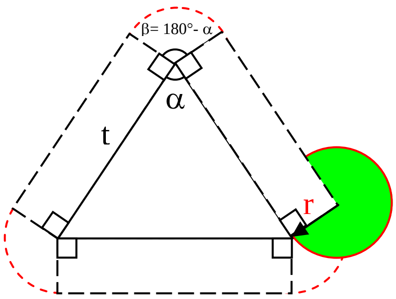
First, we will examine the case where a circle of radius *r* rolls around a square.

Let the square's side length be *s* and the length of the locus be l_1.



The straight parts of the locus are parallel to a side of the square. The corner arcs are quadrants because the circle stays a fixed distance from the corner of the square about which it is turning.
$\therefore l = \mathbf{4s+2\pi r}$.

Now, we will examine the locus, length l_2, when the circle of radius *r* turns around an equilateral triangle with side *t*.



The locus consists of arcs and straight lines.

The radius of a circle is perpendicular to any tangent of the circle. So in this case, the triangle's sides are always the tangents to the moving circle, hence the right angles in the diagram. Referring to the diagram, assume the angle of the triangle is α (= 60° in this case), and the angle of the circular arc is β (=120°), then

$$\alpha+\beta = 180°$$

As β is also equal to the exterior angle of α, so the sum of the angles of the three β's must be equal to 360°, i.e. the three arcs will form a complete circle and its circumference is equal to 2πr. The straight parts of the locus are parallel to and the same length as the triangle's sides.

$\therefore$ $\boldsymbol{l_2 = 3t + 2\pi r}$

The above argument can be extended to any **convex** polygon with any number of sides, because the exterior angles of any polygon add up to 360° so the arcs on the locus of the centre of the circle's path always add up to the circumference of the circle $2\pi r$. So if the perimeter of the polygon is *p*, the total length of the locus *l* will be

$\boldsymbol{l = p + 2\pi r}$

The argument can be further extended to any convex shape.

We now find the general rule for the length of the locus *l* when a circle rolls around any convex shape, with perimeter *p*.

The locus is in general made up of the motion of the centre along the shape plus rotation. The first part is always equal to the perimeter of the shape. Because the exterior angles of any shape add up to 360° so the arcs on the locus of the centre of the circle's path, regardless of the shape around which it is rolling, always add up to the circumference of the circle ($2\pi r$).

$\therefore$ $\boldsymbol{l = p + 2\pi r}$