Quad in quad

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Is the area of $PQRS$ always the same fraction of the area of $ABCD$? What is the size of this fraction?

1 Picture

At the beginning the best option is to make a detailed sketch of quadrilateral $ABCD$ and a smaller quadrilateral $PQRS$, which is marked correctly. The midpoints of the sides of a larger quadrilateral are vertexes of $PQRS$.

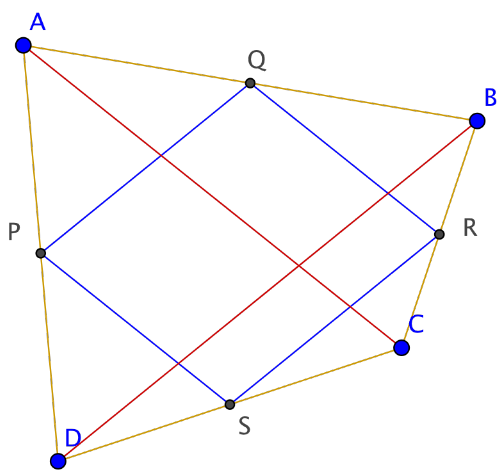

Figure 1: *Faithful* sketch

2 First findings

2.1 Triangles CRS and CBD

Triangles CRS andCBD are interesting; because the points S and R are the midpoints of the sides of a larger quadrilateral, and they are the midpoints of the triangle CBD and therefore the ratio is:

$$\frac{|DC|}{|CS|} = \frac{|CB|}{|CR|} = 2$$

which means the triangles CRS and CBD are similar.

$$|DB| : |SR| = 2$$

We also observe that the line segments SR in DB are parallel.

2.2 Triangles APQ in ADB

Deduction is the same as above.

$$\frac{|AD|}{|AP|} = \frac{|AB|}{|AQ|} = \frac{|DB|}{|PQ|} = 2$$

Therefore the line segments PQ and DB are parallel, which is also true that line segments PQ and SR are parallel, as well.

2.3 Triangles DSP in DCA

$$\frac{|DC|}{|DS|} = \frac{|DA|}{|DP|} = \frac{|AC|}{|PS|} = 2$$

In triangles DSP in DCA the line segments PS and AC are surprisingly parallel.

2.4 Triangles BQR in BAC

$$\frac{|BQ|}{|BA|} = \frac{|BR|}{|BC|} = \frac{|QR|}{|AC|} = 2$$

Line segments QR are AC parallel, which is equally true for segments QR and PS.

3 Quadrilateral PQRS is a parallelogram

$PQRS$ is a parallelogram, because the segments PS and QR are parallel, which is similar for PQ and SR.

4 Areas

Diagonals divide the quadrilateral $ABCD$ into four *large* triangles; let's have a look at that type triangle near the point R. The triangle is divided into two *small* triangles and one *small* parallelogram. Making *a'* and *b'* shorter sides of *small parallelogram*, shorter sides of the two *small* triangles are the same length as a' and b' because of the similar triangles APQ and ADB (or BQR and BAC). The area of both *small* triangles is equal to the area of *small* parallelogram. If we aren't sure about this, **triangle area formula** can be used:

$$S = \frac{1}{2} a' \cdot b' \cdot \sin\alpha$$

We can claim similarly about three other *large* triangles.

It is now obvious that the area of triangles CRS, BQR, APQ and DSP is equal to the area of parallelogram $PQRS$; in other words – the area of parallelogram $PQRS$ is a half of the area of the quadrilateral $ABCD$.

The procedure with a concave quadrilateral is the same (similar triangles). ★