We have $f_0(x) = \frac{1}{1+x^2}$, $f_{n+1}(x) = \frac{d f_n(x)}{dx}$ for $n = 0, 1, 2, \dots$.

These kinds of recursive identity are a good hint that induction might be useful.

so $f_1(x) = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$

$= \frac{-2x}{(1+x^2)^2}$.

$f_2(x) = \frac{-2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}$

Want to show that for $n \geq 1$, $(1+x^2) f_{n+1}(x) + 2(n+1)x f_n(x) + n(n+1) f_{n-1}(x) = 0$, using induction.

Base case: $n = 1$.

$(1+x^2) f_2 + 2(1+1)x f_1 + 1(1+1) f_0 = -\frac{2}{1+x^2} + \frac{8x^2}{(1+x^2)^2} - \frac{8x^2}{(1+x^2)^2} + \frac{2}{1+x^2}$

$= 0$ as required.

Now assume claim is true for $n = k$

so $(1+x^2) f_{k+1} + 2(k+1) x f_k + k(k+1) f_{k-1} = 0$.

Differentiating, $2x f_{k+1} + (1+x^2) \frac{d}{dx} f_{k+1} + 2(k+1) f_k + 2(k+1) x \frac{df_k}{dx}$

$+ k(k+1) \frac{df_{k-1}}{dx} = 0$.

$\Rightarrow (1+x^2) f_{k+2} + [2 + 2(k+1)] x f_{k+1} + [2(k+1) + k(k+1)] f_k = 0$.

$\Rightarrow (1+x^2) f_{k+2} + 2(k+2) x f_{k+1} + (k+1)(k+2) f_k = 0$.

So the claim is true for $n = k+1$ if true for $n = k$.

Hence claim is true by induction.

We have $P_n(x) = (1+x^2)^{n+1} f_n(x)$ $n = 0, 1, \dots$

$P_0(x) = (1+x^2) \cdot \frac{1}{1+x^2} = 1$.

$P_1(x) = (1+x^2)^2 \cdot \frac{-2x}{(1+x^2)^2} = -2x$.

$$P_2(x) = (1+x^2)^3\left[\frac{-2}{(1+x^2)^2} + \frac{8x^2}{(1+x^2)^3}\right]$$

$$= -2(1+x^2) + 8x^2$$

$$= 6x^2 - 2.$$

$$\frac{dP_n}{dx} = \frac{d}{dx}\left[(1+x^2)^{n+1} f_n(x)\right]$$

$$= 2(n+1)x(1+x^2)^n f_n(x) + (1+x^2)^{n+1} f_{n+1}(x).$$

So, $$P_{n+1}(x) - (1+x^2)\frac{d\,P_n(x)}{dx} + 2(n+1)x\,P_n(x) = (1+x^2)^{n+2} f_{n+1}$$

$$-(1+x^2)\left[2x(n+1)(1+x^2)^n f_n + (1+x^2)^{n+1} f_{n+1}(x)\right]$$

$$+ 2(n+1)x(1+x^2)^{n+1} f_n.$$

$$= (1+x^2) f_{n+1} - 2x(n+1) f_n - (1+x^2) f_{n+1}(x) + 2(n+1)x f_n$$

$$= 0.$$

Want to show that P_n is a polynomial of degree n, by induction.

Base case: $n=0$ P_0 is a polynomial of degree 0, as required.

In this case, you need to prove it for $n \geq 0$. Pick base case wisely!

Inductive step: assume claim is true for $n=k$, so P_k is a polynomial of degree k.

Then $\frac{dP_k}{dx}$ is a polynomial of degree $k-1$.

So $(1+x^2)\frac{dP_k}{dx}$ is a polynomial of degree $k+1$.

And $2(k+1)x\,P_k$ is a polynomial of degree $k+1$.

Write $P_k = a_k x^k + \dots + a_0$.

So $(1+x^2)\frac{dP_k}{dx}$ has leading order term $a_k x^{k+1}$.

Important to check, otherwise you might have P_{k+1} a polynomial of degree k...

and $2(k+1)x\,P_k$ has leading order term $2(k+1)a_k x^{k+1} \neq a_k x^{k+1}$. (leading order terms do not cancel)

and $P_{k+1} = (1+x^2)\frac{dP_k}{dx} - 2(k+1)x\,P_k$ is therefore a polynomial of degree $k+1$, so Claim is true for $n=k+1$ if it is true for $n=k$.

Hence, by induction, P_n is a polynomial of degree n for all $n \geq 0$.