(i) We have $C = \frac{1}{n+1} \sum_{k=1}^{n+1} x_k$

$= \frac{1}{n+1}\left[\sum_{k=1}^{n} (x_k) + x_{n+1}\right]$ Make sure it's clear what exactly is being summed

$= \frac{1}{n+1}\left[nA + x_{n+1}\right].$

(ii) We have $B = \frac{1}{n} \sum_{k=1}^{n} (x_k - A)^2.$

$= \frac{1}{n}\left[\sum_{k=1}^{n} x_k^2 - \sum_{k=1}^{n} 2Ax_k + \sum_{k=1}^{n} A^2\right]$

$= \frac{1}{n}\left[\sum_{k=1}^{n} x_k^2 - 2A\sum_{k=1}^{n} x_k + \sum_{k=1}^{n} A^2\right]$

$= \frac{1}{n}\left[\sum_{k=1}^{n} (x_k^2) - 2nA^2 + nA^2\right]$

$= \frac{1}{n}\sum_{k=1}^{n} (x_k^2) - A^2.$

(iii) We have $D = \frac{1}{n+1} \sum_{k=1}^{n+1} (x_k - C)^2$

$= \frac{1}{n+1}\left[\sum_{k=1}^{n+1} x_k^2 - 2C\sum_{k=1}^{n+1} x_k + \sum_{k=1}^{n+1} C^2\right]$

$= \frac{1}{n+1}\sum_{k=1}^{n+1} (x_k^2) - 2C^2 + C^2$

$= \frac{1}{n+1}\left[\sum_{k=1}^{n} (x_k^2) + x_{n+1}^2\right] - C^2.$

$= \frac{1}{n+1}\left[n(B+A^2) + x_{n+1}^2\right] - C^2.$

Now $C^2 = \dfrac{n^2A^2 + 2nAx_{n+1} + x_{n+1}^2}{(n+1)^2}.$

so $D = \frac{1}{(n+1)^2}\left[n(n+1)(B+A^2) + (n+1)x_{n+1}^2 - n^2A^2 - 2nAx_{n+1} - x_{n+1}^2\right].$

$= \frac{1}{(n+1)^2}\left[nA^2 + n(n+1)B + nx_{n+1}^2 - 2nA_{n+1}\right]$

$$D = \frac{n}{(n+1)^2}\left[(n+1)B + (A - x_{n+1})^2\right].$$ ← A nice neat form to work with.

—

$$\Rightarrow (n+1)D = \frac{n}{n+1}\left[(n+1)B + (A - x_{n+1})^2\right]$$

$$= nB + \underbrace{\frac{n}{n+1}(A - x_{n+1})^2}_{\geq 0 \text{ since } \frac{n}{n+1} \geq 0 \text{ as } n \in \mathbb{N} \text{ and } (A - x_{n+1})^2 \geq 0.}.$$ ← squares are very useful for inequalities.

So $(n+1)D \geq nB$, for all values of x_{n+1}.

—

Often a good strategy for these kinds of proofs about inequalities

Consider $D - B = \frac{n}{n+1}B + \frac{n}{(n+1)^2}(A - x_{n+1})^2 - B$

$$= \frac{n}{(n+1)^2}(A - x_{n+1})^2 - \frac{B}{n+1}.$$

You need the implication in both directions since the question asks you to prove 'if and only if'. (You could write out the proof in both directions, but obviously this is quicker!).

Now $D < B \iff D - B < 0$.

$$\iff \frac{n}{(n+1)^2}(A - x_{n+1})^2 - \frac{B}{n+1} < 0.$$

$$\iff (A - x_{n+1})^2 < \frac{B(n+1)}{n}.$$

$$\iff -\sqrt{\frac{B(n+1)}{n}} < A - x_{n+1} < \sqrt{\frac{B(n+1)}{n}}$$

$$\iff A - \sqrt{\frac{B(n+1)}{n}} < x_{n+1} < A + \sqrt{\frac{B(n+1)}{n}}.$$