2010 STEP III Question 8

For $P(x) = Q(x)R'(x) - Q'(x)R(x)$,

by the Quotient Rule, $\displaystyle\int \frac{P(x)}{(Q(x))^2}\,dx = \frac{R(x)}{Q(x)} + \text{constant}.$

(i). Write $\displaystyle I = \int \frac{5x^2-4x-3}{(1+2x+3x^2)^2}\,dx$

so we have $P(x) = 5x^2-4x-3$

$Q(x) = (1+2x+3x^2)$.

Need to find $R(x)$ such that $5x^2-4x-3 = Q(x)R'(x) - Q'(x)R(x)$.

Write $R(x) = a+bx+cx^2$. $\qquad Q'(x) = 2+6x$.

$R'(x) = b+2cx$.

so $Q(x)R'(x) - Q'(x)R(x) = (3x^2+2x+1)(2cx+b) - (6x+2)(a+bx+cx^2)$.

$= 6\cancel{cx^3} + 3bx^2 + 4cx^2 + 2bx + 2cx + b$

$\quad - [6\cancel{cx^3} + 6bx^2 + 2cx^2 + 6ax + 2bx + 2a]$

$= 5x^2-4x-3$.

Comparing coefficients: const: $b-2a=-3$ (1)

x: $2b+2c-6a-2b=-4$

$\Rightarrow 2c-6a=-4$ (2)

x^2: $3b+4c-6b-2c=5$

$\Rightarrow -3b+2c=5$ (3).

One solution to these simultaneous equations is

$a=0,\ b=-3,\ c=-2$ ← Due to dependence of (1), (2), (3), you can pick an a and this fixes b and c.

so $\displaystyle I = \frac{-2x^2-3x}{1+2x+3x^2} + \text{const}.$

But the solution is not unique because

$(3) + 3\cdot(1) \Rightarrow -3b+2c+3b-6a = 5-9$

$\Rightarrow 2c-6a=-4$, which is the same as (2).

So the three equations in the three variables are not independent.

Another solution is $a = 1,\ b = -1,\ c = 1$.

This gives $I = \dfrac{1 - x + x^2}{1 + 2x + 3x^2} + \text{const}$

$$= \frac{(1+2x+3x^2) - 3x - 2x^2}{1+2x+3x^2} + \text{const} \quad (*)$$

$$= 1 \quad \frac{-2x^2 - 3x}{1 + 2x + 3x} + \text{const}$$

So the two solutions are the same up to a constant. This seems like a sensible answer, which partly motivated using the trick at $(*)$.

(ii) We have $\dfrac{dy}{dx} + \dfrac{\sin x - 2\cos x}{1 + \cos x + 2\sin x}\, y = \dfrac{5 - 3\cos x + 4\sin x}{1 + \cos x + 2\sin x}$. $(*)$

Multiply by integrating factor $e^{\int \frac{\sin x - 2\cos x}{1 + \cos x + 2\sin x}\, dx}$

$$= e^{-\ln(1 + \cos x + 2\sin x)}$$

$$= e^{\ln(1 + \cos x + 2\sin x)^{-1}}$$

$$= \frac{1}{1 + \cos x + 2\sin x}$$

So $(*)$ becomes $\dfrac{d}{dx}\left(\dfrac{y}{1 + \cos x + 2\sin x}\right) = \dfrac{5 - 3\cos x + 4\sin x}{(1 + \cos x + 2\sin x)^2}$.

integrating, $\Rightarrow \dfrac{y}{1 + \cos x + 2\sin x} = \displaystyle\int \frac{5 - 3\cos x + 4\sin x}{(1 + \cos x + 2\sin x)^2}\, dx.$

As in part (i), we want $P(x) = 5 - 3\cos x + 4\sin x$
for $Q(x) = 1 + \cos x + 2\sin x$.

Write $R = a + b\cos x + c\sin x$

$R' = -b\sin x + c\cos x$

$Q' = 2\cos x - \sin x$.

So $QR' - Q'R = (1 + \cos x + 2\sin x)(-b\sin x + c\cos x) - (2\cos x - \sin x)(a + b\cos x + c\sin x)$

$$= c\cos^2 x - 2b\sin^2 x + (2c - b)\sin x\cos x - b\sin x + c\cos x$$
$$- [2b\cos^2 x - c\sin^2 x + (2c - b)\sin x\cos x - a\sin x + 2a\cos x]$$

$$= c - 2b + (a - b)\sin x + (c - 2a)\cos x.$$

And $QR' - Q'R = 5 - 3\cos x + 4\sin x$.

Comparing coefficients,

const: $c - 2b = 5$

$\cos x$: $c - 2a = -3$

$\sin x$: $a - b = 4$.

One solution is given by $a = 4, b = 0, c = 5$. ← Again, solution is not unique.

So
$$\frac{y}{1+\cos x + 2\sin x} = \frac{4 + 5\sin x}{1 + \cos x + 2\sin x} + k \qquad k = \text{const}.$$

$$= 4 + 5\sin x + k\underbrace{(1 + \cos x + 2\sin x)}_{\text{Don't forget this!}}.$$