2010 STEP II Question 1.

This question looks a bit intimidating, but it's just algebra once you've got your head around what it's asking. You just need to find a circle that passes through P, and has $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ equal to that of the curve at P.

The osculating circle has equation $(x-a)^2+(y-b)^2=r^2$ (1) for some a, b, r.

Differentiating, $2(x-a)+2(y-b)\frac{dy}{dx}=0$. $\Rightarrow \frac{dy}{dx}=\frac{a-x}{y-b}$. (2)

Differentiating again, $2+2\left(\frac{dy}{dx}\right)^2+2(y-b)\frac{d^2y}{dx^2}=0$. (3).

Curve C has equation $y=1-x+\tan x$.

at P, $x=\pi/4$ so $y=1-\frac{\pi}{4}+\tan\frac{\pi}{4}=2-\frac{\pi}{4}$.

$\frac{dy}{dx}=-1+\sec^2 x. = \tan^2 x.$

$\frac{d^2y}{dx^2}=2\tan x\sec^2 x.$

so at P, $\frac{dy}{dx}=1$

$\frac{d^2y}{dx^2}=2\tan\pi/4(1+\tan^2\pi/4)$

$=4.$

Substituting into (3),

$2+2\cdot1^2+2(2-\pi/4-b)\cdot4=0.$

$\Rightarrow$ $4+16-2\pi-8b=0$.

$\Rightarrow 8b=20-2\pi$

$\Rightarrow b=\frac{5}{2}-\frac{\pi}{4}$.

substituting into (2), $1=\frac{a-\pi/4}{2-\pi/4-5/2+\pi/4}$

$\Rightarrow a-\frac{\pi}{4}=-\frac{1}{2}$

$\Rightarrow a=\pi/4-1/2$.

Substituting into (1),

$$(\pi/4 - \pi/4 + 1/2)^2 + (2 - \pi/4 - 5/2 + \pi/4)^2 = r^2.$$

$$\Rightarrow \frac{1}{4} + \frac{1}{4} = r^2$$

$$\Rightarrow r^2 = \frac{1}{2}.$$

$$\Rightarrow r = \frac{\sqrt{2}}{2}.$$

So the osculating circle has centre $(\pi/4 - 1/2, 5/2 - \pi/4)$ and radius $\frac{\sqrt{2}}{2}$.