we have $\cos 3x = \cos(2x + x)$ ← This is sometimes a useful idea for trigonometric identities.

$$= \cos 2x \cos x - \sin 2x \sin x$$

$$= (2\cos^2 x - 1)\cos x - 2\sin x \cos x \cdot \sin x$$

$$= 2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x$$

$$= 2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x$$

$$= 4\cos^3 x - 3\cos x \quad \checkmark.$$

Likewise, $\sin 3x = \sin(2x + x)$

$$= \sin 2x \cos x + \sin x \cos 2x$$

$$= 2\sin x \cos x \cdot \cos x + \sin x(1 - 2\sin^2 x).$$

$$= 2\sin x(1 - \sin^2 x) + \sin x - 2\sin^3 x.$$

$$= 2\sin x - 2\sin^3 x + \sin x - 2\sin^3 x.$$

$$= 3\sin x - 4\sin^3 x.$$

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i. $I(\alpha) = \int_0^\alpha 7\sin x - 8\sin^3 x \, dx.$

$$= \int_0^\alpha \sin x + (6\sin x - 8\sin^3 x)\, dx$$

$$= \int_0^\alpha \sin x + 2\sin 3x \, dx.$$

$$= \left[-\cos x - \frac{2}{3}\cos 3x\right]_0^\alpha$$

$$= -c - \frac{2}{3}\cos 3x + 1 + \frac{2}{3}$$

$$= -\frac{2}{3}(4c^3 - 3c) - c + \frac{5}{3}$$

$$= -\frac{8}{3}c^3 + c + \frac{5}{3}.$$

when $c = 1$, we have $I(\alpha) = -\frac{8}{3} + 1 + \frac{5}{3} = 0.$

ii. Write $J(\alpha)$ for Eustace's calculated value of $I(\alpha)$.

$$J(\alpha) = \left[\frac{7\sin^2 x}{2} - \frac{8\sin^4 x}{4} \right]_0^\alpha$$

Careful! It's easy to forget just how useless Eustace is. He can't even integrate $\sin x$...

$$= \frac{7}{2}\sin^2\alpha - 2\sin^4\alpha$$

$$= \frac{7}{2}(1-c^2) - 2(1-c^2)^2$$

$$= \frac{7}{2} - \frac{7}{2}c^2 - 2(1-2c^2+c^4)$$

$$= \frac{3}{2} + \frac{1}{2}c^2 - 2c^4.$$

so $J(\alpha) = I(\alpha)$ when

We could deal with the case $c = -1/6$ separately. to show that $J(\alpha) = I(\alpha)$, but it's quicker and equally valid not to.

$$-\frac{8c^3}{3} + c + \frac{5}{3} = \frac{3}{2} + \frac{1}{2}c^2 - 2c^4$$

$$\Rightarrow 2c^4 - \frac{8}{3}c^3 - \frac{1}{2}c^2 + c + \frac{1}{6} = 0.$$

$$\Rightarrow 12c^4 - 16c^3 - 3c^2 + 6c + 1 = 0.$$

$$\Rightarrow (6c+1)(2c^3 - 3c^2 + 1) = 0$$

We know that $-1/6$ is a root so we know $(6c+1)$ is a factor without doing any work.

$$\Rightarrow (6c+1)(c-1)(2c^2 - c - 1) = 0$$

$$\Rightarrow (6c+1)(c-1)^2(2c+1) = 0.$$

So $c = -1/6$, $c = 1$, $c = -1/2$.

So Eustace obtains the correct value of $I(\alpha)$ when:

$\cos\alpha = -1/6$ i.e. $\alpha = \cos^{-1}(-1/6) + 2n\pi$ $n \in \mathbb{Z}$

$\cos\alpha = 1$ i.e. $\alpha = 2n\pi$.

$\cos\alpha = -1/2$ i.e. $\alpha = 2n\pi \pm \frac{2\pi}{3}$.

The question wants all values of α for which this is true - don't forget the extra multiples of 2π!