2010 STEP II Question 4.

i. We have $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$ Try the substitution $x = a-y$.

so $\frac{dx}{dy} = -1$.

limits: $x=a \Rightarrow y=0$, $x=0 \Rightarrow y=a$.

$$= -\int_a^0 \frac{f(a-y)}{f(a-y)+f(y)}\,dy$$

$$= \int_0^a \frac{f(a-y)}{f(a-y)+f(y)}\,dy. \leftarrow$$ Note that we've shown what we need to – y is only a dummy variable

So we have $I = \int_0^a \frac{f(x)}{f(x)+f(a-x)}\,dx$ (1) $\quad I = \int_0^a \frac{f(a-x)}{f(x)+f(a-x)}\,dx.$ (2)

$$(1)+(2) \Rightarrow 2I = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}\,dx$$

$$= \int_0^a 1\,.dx$$

$$= a.$$

$$\Rightarrow I = \frac{a}{2}.$$

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$$\int_0^1 \frac{\ln(x+1)}{\ln(2+x-x^2)}\,dx = \int_0^1 \frac{\ln(x+1)}{\ln(x+1)(2-x)}\,dx$$

$= I$ Using earlier part of question with $f(x) = \ln(x+1)$, $a = 1$.

$$= \frac{1}{2}.$$

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$$\int_0^{\pi/2} \frac{\sin x}{\sin(x+\pi/4)}\,dx = \int_0^{\pi/2} \frac{\sin x}{\sin x \cos \pi/4 + \sin \pi/4 \cos x}\,dx.$$

$$= \int_0^{\pi/2} \frac{\sqrt{2}\sin x}{\sin x + \cos x}\,dx$$

$$= \sqrt{2}\int_0^{\pi/2} \frac{\sin x}{\sin x + \sin(\pi/2 - x)}\,dx$$

$= \sqrt{2}\, I$ where $f(x) = \sin x$, $a = \pi/2$.

$$= \frac{\pi\sqrt{2}}{4}.$$

ii.

Write $J = \int_{1/a}^{a} \frac{f(x)}{x(f(x)+f(1/x))}\,dx$. (3) ← This seems like a sensible thing to try, given the form of the integrand we're asked to evaluate, and comparing with part (ii).

Use the substitution $y = \frac{1}{x}$

so $\frac{dx}{dy} = \frac{-1}{x^2}$.

Limits: $x = a \Rightarrow y = 1/a$, $x = 1/a \Rightarrow y = a$.

so $J = \int_{a}^{1/a} \frac{f(1/y)}{1/y\,(f(1/y)+f(y))} \cdot \frac{-1}{y^2}\,dy$.

$= -\int_{a}^{1/a} \frac{f(1/y)}{y(f(1/y)+f(y))}\,dy$.

$= \int_{1/a}^{a} \frac{f(1/y)}{y(f(1/y)+f(y))}\,dy$. (4)

so (3) + (4) $\Rightarrow 2J = \int_{1/a}^{a} \frac{f(x)+f(1/x)}{x(f(x)+f(1/x))}\,dx$

$= \int_{1/a}^{a} \frac{1}{x}\,dx$

$= [\ln x]_{1/a}^{a}$

$= \ln a - \ln(\frac{1}{a})$

$= \ln a + \ln a$

$= 2\ln a$.

$\Rightarrow J = \ln a$.

so with $f(x) = \sin x$, $a = 2$, this becomes

$J = \int_{1/2}^{2} \frac{\sin x}{x(\sin x + \sin 1/x)}\,dx$

$= \ln 2$.