2010 STEP II Q5

$A : \underline{i} + \underline{j} + \underline{k}$, $B : 5\underline{i} - \underline{j} - \underline{k}$, $2\alpha = \angle AOB$.

Then $\cos 2\alpha = \dfrac{\underline{A} \cdot \underline{B}}{|\underline{A}||\underline{B}|} = \dfrac{1\cdot 5 - 1\cdot 1 - 1\cdot 1}{\sqrt{3}\cdot\sqrt{27}} = \dfrac{3}{\sqrt{81}} = \boxed{\dfrac{1}{3}}$.

standard formula

i) L_1 has equation $\underline{r} = \lambda(m\underline{i} + n\underline{j} + p\underline{k})$.

Then L_1 is equally inclined to OA and to OB if $\angle L_1 OA = \angle L_1 OB$.

$$\Leftrightarrow \quad \frac{L_1 \cdot A}{|L_1||A|} = \frac{L_1 \cdot B}{|L_1||B|}$$

$$\Leftrightarrow \quad \frac{m+n+p}{\cancel{\sqrt{m^2+n^2+p^2}}\cdot\sqrt{3}} = \frac{5m-n-p}{\cancel{\sqrt{m^2+n^2+p^2}}\cdot\sqrt{27}}$$

$$\Leftrightarrow \quad 3(m+n+p) = 5m-n-p$$

$$\Leftrightarrow \quad 3m+3n+3p = 5m-n-p$$

$$\Leftrightarrow \quad \underline{m = 2(n+p)}.$$

If L_1 is the angle bisector of $\angle AOB$, then we must have $\angle L_1 OA = \angle L_1 OB = \alpha$, where $\cos 2\alpha = \frac{1}{3}$.

Then $\cos 2\alpha = 2\cos^2\alpha - 1 \Rightarrow \frac{1}{3} = 2\cos^2\alpha - 1 \Rightarrow \cos\alpha = \frac{\sqrt{2}}{\sqrt{3}}$.

So consider $\cos(\angle L_1 OA) = \cos\alpha$ and then use $m = 2(n+p)$:

working out this instead of $\angle L_1 OB$ will come in handy later.

$$\frac{m+n+p}{\sqrt{m^2+n^2+p^2}\cdot\sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$\Leftrightarrow \quad m+n+p = \sqrt{2}\cdot\sqrt{m^2+n^2+p^2}$

$\Leftrightarrow \quad (m+n+p)^2 = 2(m^2+n^2+p^2)$

$\Leftrightarrow \quad m^2+n^2+p^2+2(mn+mp+np) = 2(m^2+n^2+p^2)$

$\Leftrightarrow \quad m^2+n^2+p^2 = 2m(n+p)+2np \qquad (*)$

$\Leftrightarrow \quad \cancel{4(n+p)^2}+n^2+p^2 = \cancel{4(n+p)^2}+2np$

$\Leftrightarrow \quad n^2-2np+p^2=0$

$\Leftrightarrow \quad (n-p)^2=0$

$\Leftrightarrow \quad \underline{n=p}$.

So then $m=4n$.

So L_1 is the angle bisector for $\begin{pmatrix} m \\ n \\ p \end{pmatrix} = \lambda \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$, any $\lambda \neq 0$.

---

ii) L_2 has equation $\underline{r} = \mu(u\underline{i}+v\underline{j}+w\underline{k})$.

$\angle L_2OA = \alpha$. But we have already worked this out! See $(*)$:

so $\underline{u^2+v^2+w^2 = 2(uv+uw+vw)}$.

Saves you some work if you spot this

So then $x^2+y^2+z^2 = 2(yz+zx+xy)$ describes all lines which are at an angle α to OA, passing through O. So the surface is a double cone: