2010 STEP II Q7

i) $y = x^3 - 3qx - q(1+q)$ , $q>0$, $q \neq 1$.

Then $\frac{dy}{dx} = 3x^2 - 3q = 3(x - \sqrt{q})(x + \sqrt{q})$

So the curve has turning points at $x = \pm \sqrt{q}$.

At $x = q^{\frac{1}{2}}$, $y = q^{\frac{3}{2}} - 3q \cdot q^{\frac{1}{2}} - q(1+q)$

$= -2q^{\frac{3}{2}} - q(1+q)$

$< 0$ for $q > 0$. ← since all terms are strictly negative

At $x = -q^{\frac{1}{2}}$, $y = q^{-\frac{3}{2}} + 3q \cdot q^{\frac{1}{2}} - q(1+q)$

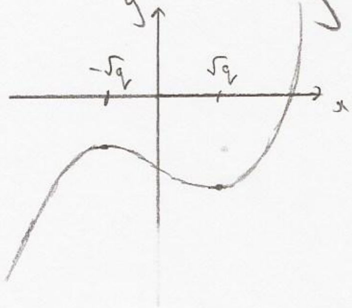
$= 2q^{\frac{3}{2}} - q(1+q)$

$= -q(q - 2q^{\frac{1}{2}} + 1)$

$= -q(q^{\frac{1}{2}} - 1)^2$

$< 0$ for $q > 0$.

So both turning points are below the $x$-axis, so the cubic curve will only cross the $x$-axis once.



ii) $x^3 - 3qx - q(1+q) = 0$

$x = u + \frac{q}{u} \Rightarrow ux = u^2 + q$

$\Rightarrow u^3x^3 = u^6 + 3qu^4 + 3q^2u^2 + q^3$

$x^3 - 3qx - q(1+q) = 0$

multiplying through by $u^3$ makes the calculation clearer, but it is not necessary.

$\Rightarrow u^3x^3 - 3qu^3x - q(1+q)u^3 = 0$

$\Rightarrow u^6 + \cancel{3qu^4} + \cancel{3q^2u^2} + q^3 - 3q(\cancel{u^4} + \cancel{qu^2}) - q(1+q)u^3 = 0$

$\Rightarrow u^6 - q(1+q)u^3 + q^3 = 0$

$\Rightarrow (u^3)^2 - q(1+q)(u^3) + q^3 = 0.$

So $u^3$ satisfies the quadratic equation $t^2 - q(1+q)t + q^3 = 0$.

This has solution $t = \dfrac{q(1+q) \pm \sqrt{q^2(1+q)^2 - 4q^3}}{2}$

$= \frac{1}{2}q(1+q) \pm \frac{1}{2}\sqrt{q^4 - 2q^3 + q^2}$

$= \frac{1}{2}q(1+q) \pm \frac{1}{2}q(q-1)$

So $t = \frac{1}{2}q(1+q) + \frac{1}{2}q(q-1) = \frac{1}{2}\cdot q \cdot 2q = q^2$ or $t = \frac{1}{2}q(1+q) - \frac{1}{2}q(q-1) = \frac{1}{2}q \cdot 2 = q$.

So $u^3 = q$ or $q^2 \Rightarrow u = q^{1/3}$ or $q^{2/3}$

Then $x = u + \frac{q}{u} \Rightarrow x = q^{1/3} + q^{2/3}$ or $q^{2/3} + q^{1/3}$ ← so the two solutions are equal!

So $\underline{x = q^{1/3} + q^{2/3}}$ is the real root of the cubic equation.

iii) $t^2 - pt + q = 0$ has roots $\alpha$ and $\beta$.

Then $p = \alpha + \beta$ and $q = \alpha\beta$.

Then $(\alpha+\beta)^3 = \alpha^3 + \beta^3 + 3\alpha^2\beta + 3\alpha\beta^2$ ← expand a general expression and then substitute in what you know

$= \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$

$\Rightarrow p^3 = \alpha^3 + \beta^3 + 3qp$.

$\Rightarrow \underline{\alpha^3 + \beta^3 = p^3 - 3qp}$ as required.

If one root is the square of the other, then either $\alpha^2 = \beta$ or $\beta^2 = \alpha$.

Either way, $(\alpha^2 - \beta)(\beta^2 - \alpha) = 0$. Then

$$0 = (\alpha^2 - \beta)(\beta^2 - \alpha) = \alpha^2\beta^2 - \alpha^3 - \beta^3 + \alpha\beta$$
$$= (\alpha\beta)^2 + \alpha\beta - (\alpha^3 + \beta^3)$$
$$= q^2 + q - p^3 + 3qp$$

$\Rightarrow \underline{p^3 - 3qp - q(q+1) = 0}$

Then this is in the same form as the cubic equation in part (ii), with $x = p$.

If, as before, $q > 0$, $q \neq 1$, $p$ real, then we have that

$\underline{p = q^{1/3} + q^{2/3}}$.