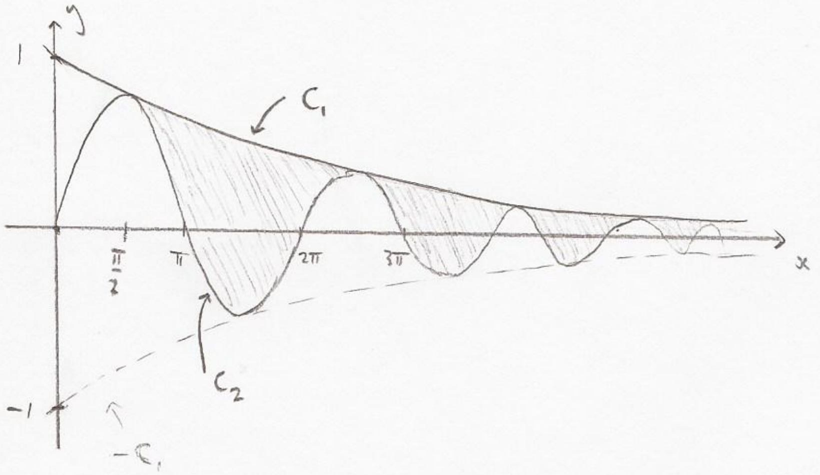
2010 STEP II Q8

$C_1 : y = e^{-x} \quad (x > 0) \quad , \quad C_2 : y = e^{-x} \sin x \quad (x > 0)$

C_1 is a standard exponential decay curve. C_2, however, oscillates as $\sin x$ takes values between -1 and 1, so C_2 oscillates between C_1 and $-C_1$.



The two curves touch at $x_1 = \frac{\pi}{2}$, $x_2 = \frac{5\pi}{2}$, ..., $x_n = (4n-3)\frac{\pi}{2}$, $x_{n+1} = (4n+1)\frac{\pi}{2}$.

Then $A_n = \int_{x_n}^{x_{n+1}} e^{-x} \, dx - \underbrace{\int_{x_n}^{x_{n+1}} e^{-x} \sin x \, dx}_{\text{call this } I_n}$.

$$\Rightarrow A_n = \left[-e^{-x} \right]_{x_n}^{x_{n+1}} - I_n .$$

$$I_n = \int_{x_n}^{x_{n+1}} e^{-x} \sin x \, dx$$

$$= \left[-e^{-x} \sin x \right]_{x_n}^{x_{n+1}} + \int_{x_n}^{x_{n+1}} e^{-x} \cos x \, dx$$

$\sin x_n = 1$ for all n

we're back to where we started!

$$= \left[-e^{-x} \right]_{x_n}^{x_{n+1}} + \left[\cancel{-e^{-x} \cos x} \right]_{x_n}^{x_{n+1}} - \int_{x_n}^{x_{n+1}} e^{-x} \sin x \, dx$$

$\cos\left(\frac{\pi}{2} + 2k\pi\right) = 0$ for all k

$$= \left[-e^{-x} \right]_{x_n}^{x_{n+1}} - I_n$$

So $I_n = \left[-e^{-x}\right]_{x_n}^{x_{n+1}} - I_n$

$$\Rightarrow \quad I_n = \frac{1}{2}\left[-e^{-x}\right]_{x_n}^{x_{n+1}}$$

So $A_n = \left[-e^{-x}\right]_{x_n}^{x_{n+1}} - I_n$

$$= \frac{1}{2}\left[-e^{-x}\right]_{x_n}^{x_{n+1}}$$ ← we left it in this form for as long as possible so we could evaluate it all in one go now.

$$= \frac{1}{2}\left(e^{-x_n} - e^{-x_{n+1}}\right)$$

$$= \frac{1}{2}\left(e^{-(4n-3)\frac{\pi}{2}} - e^{-(4n+1)\frac{\pi}{2}}\right)$$

$$= \frac{1}{2}\left(e^{2\pi} - 1\right) e^{-(4n+1)\frac{\pi}{2}}$$ as required.

Note that $A_{n+1} = \frac{1}{2}\left(e^{2\pi}-1\right)e^{-(4n+5)\frac{\pi}{2}} = e^{-2\pi} A_n$

and $A_1 = \frac{1}{2}\left(e^{2\pi}-1\right)e^{-\frac{5\pi}{2}}$.

Then $\sum_{n=1}^{\infty} A_n$ is an infinite geometric series with ratio $e^{-2\pi}$, so

$$\sum_{n=1}^{\infty} A_n = \frac{A_1}{1-e^{-2\pi}} = \frac{e^{2\pi}A_1}{e^{2\pi}-1} = \frac{\frac{1}{2}\cancel{\left(e^{2\pi}-1\right)}e^{-\pi/2}}{\cancel{e^{2\pi}-1}} = \frac{1}{2}e^{-\pi/2}.$$