2010 STEP I Question 1.

Want to find a, b, c, d such that

$$5x^2 + 2y^2 - 6xy + 4x - 4y \equiv a(x - y + 2)^2 + b(cx + y)^2 + d.$$

$$RHS \equiv a(x^2 + y^2 + 4 - 2xy + 4x - 4y) + b(c^2x^2 + 2cxy + y^2) + d.$$

Comparing coefficients:

x^2: $a + bc^2 = 5$ (1)

y^2: $a + b = 2$ (2)

xy: $-2a + 2cb = -6$ (3)

x: $4a = 4$ (4)

y: $-4a = -4$ (5)

const: $4a + d = 0$. (6).

(4) and (5) $\Rightarrow$ $a = 1$.

(2) $\Rightarrow$ $b = 1$.

(3) $\Rightarrow$ $c = -2$.

Check for consistency using (1) ← This is an important step to check that the equation actually can be written in this form!

$a + bc^2 = 1 + 1 \cdot 4 = 5$ ✓

(6) $\Rightarrow$ $d = -4$.

So $a = 1$, $b = 1$, $c = -2$, $d = -4$.

Want to solve simultaneous equations

$$5x^2 + 2y^2 - 6xy + 4x - 4y = 9 \quad (*)$$

$$6x^2 + 3y^2 - 8xy + 8x - 8y = 14 \quad (**)$$

Try rewriting (**) ← Seems like the only sensible thing to try...

$$LHS = 6x^2 + 3y^2 - 8xy + 8x - 8y \equiv a(x - y + 2)^2 + b(cx + y)^2 + d.$$

Comparing coefficients:

x^2: $a + bc^2 = 6$ (1)

y^2: $a + b = 3$ (2)

xy: $-2a + 2cb = -8$ (3)

x: $4a = 8$. (4)

y: $-4a = -8$ (5)

const: $4a + d = 0$. (6).

(4) and (5) $\Rightarrow$ $a = 2$.

(2) $\Rightarrow$ $b = 1$.

(3) $\Rightarrow$ $c = -2$

Check for consistency using (1). ← Even more important to check this time, since we only guessed that it works... And it does!

$a + bc^2 = 2 + 1 \cdot 4 = 6$ ✓

(6) $\Rightarrow$ $d = -8$.

So the simultaneous equations are:

$(x - y + 2)^2 + (-2x + y)^2 - 4 = 9$.

$2(x - y + 2)^2 + (-2x + y)^2 - 8 = 14$.

write $A = (x - y + 2)^2$ $\quad B = (-2x + y)^2$

then $\quad A + B = 13$

$2A + B = 22$

$\Rightarrow A = 9, \quad B = 4$.

So $x - y + 2 = \pm 3$, $-2x + y = \pm 2$.

Case 1: $x - y + 2 = 3$ (1)

$-2x + y = 2$. (2)

(1) + (2) $\Rightarrow$ $-x + 2 = 5$

$\Rightarrow x = -3$.

re-substitute into (2): $6 + y = 2 \Rightarrow y = -4$.

Case 2: $x - y + 2 = -3$

$-2x + y = 2$.

$\Rightarrow -x + 2 = -1$

$\Rightarrow x = 3$.

so $-6 + y = 2$

$\Rightarrow y = 8$.

case 3: $x - y + 2 = 3$

$-2x + y = -2$.

$\Rightarrow -x + 2 = 1$

$\Rightarrow x = 1$.

so $-2 + y = -2$

$\Rightarrow y = 0$.

Case 4: $x - y + 2 = -3$
$-2x + y = -2$

$\Rightarrow -x + 2 = -5$

$\Rightarrow x = 7$

so $-14 + y = -2$

$\Rightarrow y = 12.$

so solutions are:

$x = -3, y = -4$

$x = 3, y = 8$

$x = 1, y = 0$

$x = 7, y = 12.$
