2010 STEP I Q2

$$y = \left(\frac{x-a}{x-b}\right)e^x = (x-a)(x-b)^{-1}e^x$$

Differentiating using the product rule: ← it's a triple product, so be careful!

$$\frac{dy}{dx} = 1\cdot(x-b)^{-1}e^x + (x-a)\cdot\frac{d}{dx}\left[(x-b)^{-1}e^x\right]$$

$$= \frac{1}{x-b}e^x + (x-a)\left[\frac{-1}{(x-b)^2}e^x + \frac{1}{x-b}e^x\right]$$

$$= \frac{1}{(x-b)^2}e^x\left[(\cancel{x}-b) - (\cancel{x}-a) + (x-a)(x-b)\right]$$

$$= \frac{e^x}{(x-b)^2}\left[x^2 - (a+b)x + ab + a - b\right].$$

Then $\frac{dy}{dx} = 0$ when $x^2 - (a+b)x + ab + a - b = 0$.

We know that the curve must have 2 stationary points, so we need the discriminant of this quadratic to be > 0: ← key idea.

$$(a+b)^2 - 4(ab + a - b) > 0$$

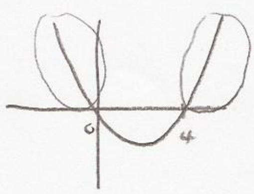
$$\Rightarrow (a+b)^2 - 4ab - 4(a-b) > 0$$

$$\Rightarrow (a-b)^2 - 4(a-b) > 0 \quad \leftarrow \text{since } (a+b)^2 - 4ab = a^2 - 2ab + b^2 = (a-b)^2$$

$$\Rightarrow (a-b)(a-b-4) > 0$$

If we let $k = a-b$, then this becomes $k(k-4) > 0$, and we can see that this holds for $k > 4$ and $k < 0$,

i.e. $a - b > 4$ or $a - b < 0$.



A sketch is very helpful to check inequalities.

i) When $a=0$, $b=\frac{1}{2}$, the expression for the stationary points of the curve becomes

$$x^2 - \frac{1}{2}x - \frac{1}{2} = 0.$$

$a-b = \frac{-1}{2} < 0$, so we expect 2 stationary points.

$$\Rightarrow 2x^2 - x - 1 = 0.$$

$$\Rightarrow (2x+1)(x-1) = 0$$

Then there are stationary points at $x = \frac{-1}{2}$ and $x = 1$.

The curve is $y = \left(\frac{x}{x-\frac{1}{2}}\right)e^x$, so the vertical asymptote is at $x = \frac{1}{2}$.

So the stationary points are on either side of this asymptote.

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Working out the y-values of the stationary points, we have

$x = \frac{-1}{2}$, $y = \frac{1}{2}e^{-1/2} = \frac{1}{2\sqrt{e}}$, $x = 1$, $y = 2e$.

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The curve only crosses the x-axis at $x=0$, since e^x is always positive, and this is also the y-intercept.

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We also note that the curve is positive just to the right of the asymptote, and negative just to the left. Also, the curve tends to $y = e^x$ as $x \to \pm\infty$.

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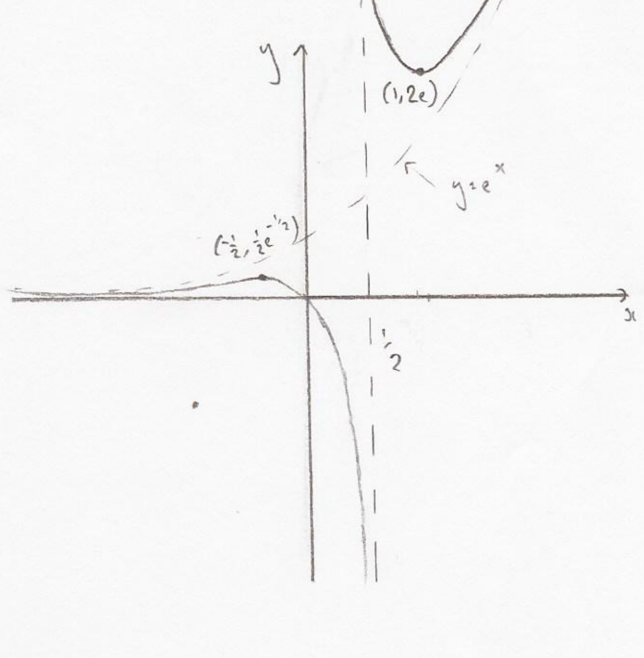
We now have enough information to sketch the curve:



ii) When $a = \frac{9}{2}$, $b = 0$, the expression for the stationary points of the curve becomes

$x^2 - \frac{9}{2}x + \frac{9}{2} = 0$ $\quad$ $a - b = \frac{9}{2} > 4$, so we expect 2 stationary points.

$\Rightarrow 2x^2 - 9x + 9 = 0$

$\Rightarrow (2x-3)(x-3) = 0$

Then there are stationary points at $x = \frac{3}{2}$ and $x = 3$.

The curve is $y = \left(\frac{x - \frac{9}{2}}{x}\right)e^x$, so the vertical asymptote is at $x = 0$.

The stationary points are $(\frac{3}{2}, -2e^{3/2})$ and $(3, \frac{-1}{2}e^3)$.

The curve crosses the x-axis at $x = \frac{9}{2}$.

The curve is negative just to the right of the asymptote, and positive just to the left.

Also, the curve tends to $y = e^x$ as $x \to \pm\infty$.

Sketching: