2010 STEP 1 QUESTION 6.

Want to show $y = e^x$ is a solution to $(*)$.

Suppose $y = e^x$. Then $\frac{dy}{dx} = \frac{d^2y}{dx^2} = e^x$.

$$\text{LHS of } (*) = (x-1)e^x - xe^x + e^x$$
$$= 0 = \text{RHS of } (*)$$

so $y = e^x$ satisfies $(*)$.

Let $y = ue^x$ for some $u = f(x)$.

Then $\frac{dy}{dx} = ue^x + e^x \frac{du}{dx}$ (using product rule)

$\frac{d^2y}{dx^2} = ue^x + 2e^x \frac{du}{dx} + e^x \frac{d^2u}{dx^2}$ (product rule again).

Substituting into $(*)$,

$$(x-1)\left[ue^x + 2e^x \frac{du}{dx} + e^x \frac{d^2u}{dx^2}\right] - x\left[ue^x + e^x \frac{du}{dx}\right] + ue^x = 0.$$

Simplifying,

$$(x-2)e^x \frac{du}{dx} + (x-1)e^x \frac{d^2u}{dx^2} = 0.$$

Now $e^x \neq 0$ for x real so this becomes

$$(x-1)\frac{d^2u}{dx^2} + (x-2)\frac{du}{dx} = 0. \quad (**).$$

Write $v = \frac{du}{dx}$.

Then $(**)$ becomes

$$(x-1)\frac{dv}{dx} + (x-2)v = 0.$$

$$\Rightarrow (x-1)\frac{dv}{dx} = (2-x)v.$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \frac{2-x}{x-1}$$

Integrating,

$$\int \frac{1}{v}\, dv = \int \frac{2-x}{x-1}\, dx.$$

so $\ln|v| = \int -1 + \frac{1}{x-1}\, dx$ ← have written $\frac{2-x}{x-1} = \frac{1+(1-x)}{x-1}$

$= -x + \ln|x-1| + c$ This is a really useful trick for integrating fractions.

$c = \text{const.}$

Exponentiate to get

$|v| = |k| e^{-x} |x-1| \qquad k = e^c.$

so take $v = ke^{-x}(x-1)$. ← Don't forget to continue...

$\Rightarrow \frac{du}{dx} = ke^{-x}(x-1).$

Integrating,

$u = k\int xe^{-x} - e^{-x}\, dx.$

$= k[-xe^{-x} - e^{-x} + e^{-x} + \text{const.}]$

$= -kxe^{-x} + l$

$l = \text{const.}$

Integrate $\int xe^{-x}dx$ by parts:

$u = x, \quad v' = e^{-x}$

$u' = 1 \quad v = -e^{-x}.$

so $\int xe^{-x} dx = -xe^{-x} + \int e^{-x} dx$

$= -xe^{-x} - e^{-x}.$

Don't forget to continue, still not done yet!

substituting into $y = ue^x$ gives

$y = -kx + le^x.$

set $A = -k$, $B = l$ to give

$y = Ax + Be^x$ a solution to $(*)$, as required.