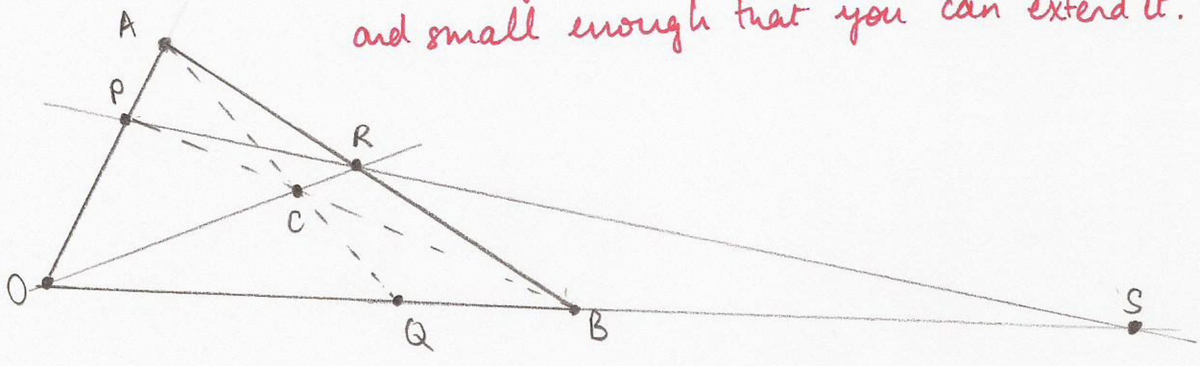
Draw a diagram!!!

Make it big enough that you can add detail inside later, and small enough that you can extend it.



$\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$.

We have P at the intersection of OA and BC. ← Alternatively we could show that $\underline{p}$ has the correct form to satisfy equations for the lines OA and BC.

P lies on OA $\Rightarrow \underline{p} = \mu \underline{a}$

P lies on BC $\Rightarrow \underline{p} = \underline{b} + \lambda(\underline{c} - \underline{b})$

$= \underline{b} + \lambda\left(\alpha \underline{a} + (\beta - 1)\underline{b}\right).$

so $\mu \underline{a} = \underline{b} + \lambda(\alpha \underline{a} + (\beta-1)\underline{b})$.

$\Rightarrow \mu = \lambda\alpha$ and $1 + \lambda(\beta - 1) = 0$. (by comparing coefficients of $\underline{a}$ and $\underline{b}$).

$\Rightarrow \lambda = \dfrac{1}{1-\beta}$.

so $\mu = \dfrac{\alpha}{1-\beta}$

and $\underline{p} = \dfrac{\alpha \underline{a}}{1-\beta}$.

Likewise, Q at intersection of OB and AC.

Q lies on OB $\Rightarrow \underline{q} = \mu \underline{b}$.

Q lies on AC $\Rightarrow \underline{q} = \underline{a} + \lambda(\underline{c} - \underline{a})$

$= \underline{a} + \lambda((\alpha - 1)\underline{a} + \beta \underline{b})$.

so $\mu \underline{b} = \underline{a} + \lambda((\alpha-1)\underline{a} + \beta\underline{b})$.

$\Rightarrow \mu = \lambda\beta, \quad 1 + \lambda(\alpha - 1) = 0.$

so $\lambda = \frac{1}{1-\alpha}$

so $\mu = \frac{\beta}{1-\alpha}$

so $\underline{q} = \frac{\beta \underline{b}}{1-\alpha}$.

Now show that $\underline{r} = \frac{\alpha \underline{a} + \beta \underline{b}}{\alpha+\beta}$ lies on OC and AB. ← Alternatively we could find the intersection of OC and AB and show that it is at R.

First, $\underline{r} = \frac{1}{\alpha+\beta} \cdot \underline{c}$ so R lies on OC.

Now on AB, $\underline{x} = \underline{a} + \lambda(\underline{b} - \underline{a})$

$= (1-\lambda)\underline{a} + \lambda \underline{b}$.

set $\lambda = \frac{\beta}{\alpha+\beta}$ so $1-\lambda = \frac{\alpha+\beta-\beta}{\alpha+\beta} = \frac{\alpha}{\alpha+\beta}$

so $\underline{r}$ satisfies $\underline{r} = (1-\lambda)\underline{a} + \lambda \underline{b}$ when $\lambda = \beta/_{\alpha+\beta}$.

so $\underline{r}$ lies on AB.

Lines OB and PR intersect at S.

Write $\underline{s}$ for position vector of S. ← Although obvious, this is worth saying since they haven't defined the notation in the question.

S lies on OB so $\underline{s} = \mu \underline{b}$.

Equation of PR: $\underline{x} = \underline{p} + \lambda(\underline{r} - \underline{p})$

$= \frac{\alpha \underline{a}}{1-\beta} + \lambda\left(\frac{\alpha \underline{a} + \beta \underline{b}}{\alpha+\beta} - \frac{\alpha \underline{a}}{1-\beta}\right)$.

s lies on PR so

$$\mu \underline{b} = \frac{\alpha \underline{a}}{1-\beta} + \lambda\left(\frac{\alpha \underline{a}}{\alpha+\beta} - \frac{\alpha \underline{a}}{1-\beta} + \frac{\beta \underline{b}}{\alpha+\beta}\right).$$

so $\frac{\alpha}{1-\beta} + \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{\alpha}{1-\beta}\right) = 0$, $\mu = \frac{\beta\lambda}{\alpha+\beta}$.

$\Rightarrow \frac{\alpha}{1-\beta} + \lambda\left(\frac{\alpha - 2\alpha\beta - \alpha^2}{(\alpha+\beta)(1-\beta)}\right) = 0$.

$\Rightarrow \lambda = \frac{\alpha+\beta}{1-2\beta-\alpha}$. $\Rightarrow \mu = \frac{\beta}{1-2\beta-\alpha}$.

so $\underline{s} = \dfrac{\beta \underline{b}}{1-2\beta-\alpha}$.

write $|\underline{a}|=a$, $|\underline{b}|=b$. ← Again, worth saying although obvious.

Now $\vec{OQ} = \dfrac{\beta \underline{b}}{1-\alpha}$

so $OQ = \dfrac{\beta b}{1-\alpha}$.

and $BQ = b - \dfrac{\beta b}{1-\alpha}$ ← B and Q collinear, the diagram is a good reminder!

$= \dfrac{(1-\alpha-\beta)b}{1-\alpha}$.

so $\dfrac{OQ}{BQ} = \dfrac{\beta b}{1-\alpha} \cdot \dfrac{1-\alpha}{(1-\alpha-\beta)b}$

$= \dfrac{\beta}{1-\alpha-\beta}$.

And $OS = \dfrac{\beta b}{|1-2\beta-\alpha|}$ ← be careful, there is no reason why we should not have $2\beta+\alpha>1$.

and $BS = \left|\dfrac{\beta b}{1-2\beta-\alpha} - b\right|$ (Although this cancels out. But that's not the point!).

$= \dfrac{|\beta+\alpha-1|\,b}{|1-2\beta-\alpha|}$

$= \dfrac{(1-\beta-\alpha)b}{|1-2\beta-\alpha|}$.

so $\dfrac{OS}{BS} = \dfrac{\beta b}{|1-2\beta-\alpha|} \cdot \dfrac{|1-2\beta-\alpha|}{(1-\beta-\alpha)b}$

$= \dfrac{\beta}{1-\alpha-\beta}$

so $\dfrac{OQ}{BQ} = \dfrac{OS}{BS}$.