2010 STEP I Q8

(i) If $a^3 + 3b^3 = 9c^3$, then $a^3 = 3(3c^3 - b^3)$, so a^3 is divisible by 3.

But then 3 is prime, so it cannot be split between the factors of a^3, so in fact we must have that a is divisible by 3.

Then $a = 3d$, for some integer d. ← key idea.

Substituting this into our original equation, we have

$(3d)^3 + 3b^3 = 9c^3$

$\Leftrightarrow 9d^3 + b^3 = 3c^3$

$\Leftrightarrow b^3 = 3(c^3 - 3d^3)$, so 3 divides b^3, so 3 divides b as before.

So $b = 3e$, some integer e.

Then $9d^3 + (3e)^3 = 3c^3$

$\Leftrightarrow 3d^3 + 9e^3 = c^3$

$\Leftrightarrow c^3 = 3(d^3 + 3e^3)$, so 3 divides c^3, so 3 divides c as before.

So $c = 3f$, some integer f.

Then $3d^3 + 9e^3 = (3f)^3$

$\Leftrightarrow d^3 + 3e^3 = 9f^3$.

Then this is the same equation we started with, so if a, b, c is an integer solution of the equation, then so is $d = \frac{a}{3}, e = \frac{b}{3}, f = \frac{c}{3}$.

We can repeat this indefinitely, so $\frac{a}{3^n}, \frac{b}{3^n}, \frac{c}{3^n}$ is an integer solution for any $n \geq 0$.

This is known as an infinite descent method.

But if $\frac{a}{3^n}$ is an integer for all $n \geq 0$, then we must have $a = 0$, and similarly for b and c.

So the only integer solution is $a = b = c = 0$.

ii) $p^4 + 2q^4 = 5r^4$

For some integer a, consider the last digit of a, a^4 and $2a^4$:

a	a^4	$2a^4$
0	0	0
1	1	2
2	6	2
3	1	2
4	6	2
5	5	0
6	6	2
7	1	2
8	6	2
9	1	2

Then if $p^4 + 2q^4 = 5r^4$, $p^4 + 2q^4$ must have last digit 5 or 0.

p^4 can have last digit 0, 1, 5 or 6, $2q^4$ can have last digit 0 or 2.

Then $p^4 + 2q^4$ ends in 5 or 0 only when p^4 ends in 5 or 0 and $2q^4$ ends in 0.

This corresponds to p ending in 5 or 0, and q ending in 5 or 0.

Hence p and q must be multiples of 5.

Then we can write $p = 5s$, $q = 5t$, some integers s and t. ← same trick as in (i)

Then $(5s)^4 + 2(5t)^4 = 5r^4$

$\Rightarrow 5^3 \cdot s^3 + 2 \cdot 5^3 \cdot t^3 = r^3$, so 5 divides r^3, so 5 divides r.

Then $r = 5u$, some integer u.

$\Rightarrow 5^3 \cdot s^3 + 2 \cdot 5^3 t^3 = 5(5u)^3$

$\Rightarrow s^3 + 2t^3 = 5u^3$

So, if p, q, r is a solution, then $\frac{p}{5}, \frac{q}{5}, \frac{r}{5}$ is also a solution.

By the previous argument, we must have $\underline{p = q = r = 0}$.