Stage 5 *

Staircase Sequences

Zach T

The first thing I noticed was :-

$$1, \quad 1+\frac{1}{1}, \quad 1+\cfrac{1}{1+\cfrac{1}{1}}, \quad 1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}, \ldots$$

The denominator in the fraction is always the preceding term.

Next I calculated the answers:

$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{8} \ldots$

It is easy to predict the next term, because this is a type of Fibonacci sequence. The numerator starts 1, 2, 3, 5, 8... the denominator starts 1, 1, 2, 3, 5... Now the numerator becomes the denominator of the next term.

If we use algebra you can see the pattern:

1	2	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{8}{5}$ ... e.t.c
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
a	b	$\frac{(a+b)}{b}$	$\frac{(a+2b)}{(a+b)}$	$\frac{(2a+3b)}{(a+2b)}$

We are calculating the Golden Ratio! Each term will get closer but will never reach it. This is an infinite series, because ϕ (phi) is an irational number.

You can make the link to Fibonacci numbers much clearer:

$1 = a$
$1 = b$
$2 = (a+b)$
$3 = (a+2b)$
$5 = (2a+3b)$
$8 = (3a+5b)$
$13 = (5a+8b) \ldots$

$$\frac{1}{1}, \quad \frac{2}{1}, \quad \frac{3}{2}, \quad \frac{5}{3}, \quad \frac{8}{5}$$

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

$$\frac{b}{a}, \quad \frac{a+b}{b}, \quad \frac{a+2b}{a+b}, \quad \frac{2a+3b}{a+2b}, \quad \frac{3a+5b}{2a+3b}$$

We can also write an expression:

$$\frac{F_n + F_{(n+1)}}{F_{(n+1)}}$$

F_n = any Fibonacci number.

$F_{(n+1)}$ = next consecutive Fibonacci number.

e.g $n = 25$

$$\frac{75{,}025 + 121{,}393}{121{,}393} = 1{\cdot}618 \text{ (3dp)}$$

Or you can also write:-

$$\frac{F_{(n+1)}}{F_n}$$

which is even easier.

The next sequence, the denominator in the fraction is always the preceding term + 1.

$$1, \xrightarrow{+1} 1+\frac{1}{2}, \xrightarrow{+1} 1+\cfrac{1}{2+\cfrac{1}{2}}, \xrightarrow{+1} 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2}}}, \ldots$$

The values are:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}$$

And it's easy to predict the next terms:

$$\frac{41}{29}, \frac{99}{70}, \frac{239}{169}$$

etc

This is because:

$$\frac{A}{B} \longrightarrow \frac{C}{D} \qquad \frac{A}{B} = T_n \qquad \frac{C}{D} = T_{(n+1)}$$

$A + B = D$

$B + D = C$ so $B + A + B = C$ $\boxed{\begin{array}{l} \text{or } C = A + 2B \\ \text{and } D = A + B \end{array}}$

If you calculate the value of the fractions in this sequence, they are all getting closer to $\sqrt{2}$, but they will never reach it because $\sqrt{2}$ is irrational so this sequence is infinite.

These ~~ty~~ types of fractions are called 'continued fractions'. I first read about these in a brilliant book 2 years ago, called 'The Number Devil' by H.M. Enzensberger. I highly recommend this to anyone who loves maths.