Deep Progress in Mathematics - The Improving Attainment in Mathematics Project

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Contents

What do we know already?

In 1976 Richard Skemp described how a 'relational' understanding of mathematics was much more powerful, long-lasting and useful than an 'instrumental' understanding.

In 1986 Brenda Denvir and Margaret Brown showed that learning mathematics is not a linear process, and that being immersed in one aspect of mathematics can frequently lead to unexpected learning.

In 1987 Afzal Ahmed published the results of a project with teachers who worked with low attaining students. They found that nearly all the students were able to use sophisticated thinking skills and learnt better if they were given time to make choices, to discuss, and to explore mathematics.

In 1997 researchers at King's College London found that teachers who made connections, 'connectionist teachers', were more successful than those who taught in a technical or fragmented way.

In 1997 Jo Boaler showed that students who learnt mathematics through open-ended exploration and problem-solving did better in tests, and other types of mathematics assessment, than similar students who were taught only to perform techniques and follow procedures.

In 1999 Carol Dweck showed that learners and teachers who believed that intelligence was flexible, and the goal was to learn as much as they can, were more successful than those who believed in finishing tasks and passing tests.

In 2003 a comparative study of seven nations found that, whatever the lesson, task or assessment style, teachers in highest achieving nations tended to focus on relationships, connections and complexities within mathematics, not reducing everything to technical performance (Hiebert et al.).

Teaching low attaining students

There are two aspects to low attainment in mathematics:

not knowing enough mathematics and not knowing how to learn mathematics

In spite of this wealth of research, it is customary for students in the lowest achieving groups to be given repetitive, procedural, fragmented, disjointed, simplified mathematics.

This booklet describes how a group of ten teachers taught low attaining groups in secondary school, and what features were seen to be important. They were neither told what to do, nor how to do it. Instead, they had a shared commitment to improving the attainment of their lowest attaining students by building on the belief that:

all students can think hard about mathematics, and thus do better at mathematics.

If you share this belief, this booklet will show you how a group of varied teachers try to put the belief into practice, in many different ways, with a range of results. If you do not share this belief, this booklet will have little to offer you.

The belief guiding the Improving Attainment in Mathematics Project (IAMP) is that ***more good can be done by helping learners develop thinking skills and understanding throughout all their mathematics lessons, than by teaching them only to perform and remember particular methods and topics. The students are prepared to learn more mathematics and work well with mathematics in later life. All students are willing and able to work hard.***

The teachers in this Project believe that students need to start secondary mathematics on a good basis. There is a public belief that these students need to 'catch-up' by rushing through earlier concepts again very quickly. This approach could, if used unwisely, lead to a diet of repeated topics and repetitive tasks. The belief in 'catch-up' is partly generated by genuine concern about students' ability to handle more advanced concepts, partly by political targets, partly by the provision to all schools of materials with which to do this task, and partly by the misinterpretation of these materials and their purpose by school managers and governors.

We know, from research, that groups of such students are often

- Given repetitive work of a kind which is very simplified.
- Offered mathematics in a step-by-step way.
- Focused on arithmetic in imaginary 'everyday' contexts.
- Expected to memorise unconnected topics and methods.
- Expected to re-do, again and again, work they have done before because they may have forgotten methods.

Project teachers believed that there were better ways of helping their students become good mathematical learners than these, which may only lead to short-term, superficial, success. Indeed, some of these students work harder than any others in school, because all work is so difficult for them.

The aim is to generate **deep progress in mathematics.**

Deep progress in mathematics

In this project we intended that those who had been placed in groups of low-attaining students would make deep progress in mathematics.

Deep progress means that students

- learn more mathematics,
- get better at learning mathematics,
- feel better about themselves as mathematics students.

Sometimes the last of these follows from the other two; sometimes students have to feel better before they learn more; sometimes students have to redevelop good learning habits before they can move on and learn more. Ideally, students will make progress in all three aspects.

When students enter secondary schools, many who have not achieved level 4[1], or have only just managed to achieve it, are already on the way to feeling that they are failing at mathematics. They may then be labelled as 'low attainers' and placed in low-achieving groups. Their past achievements have been low relative to others, yet there are many reasons why they may not have reached the coveted level 4, and none of these reasons relate to innate lack of fixed ability:

- Disrupted schooling which has led to fragmentary mathematics experience, possibly missing some key conceptual developments.
- Cultural differences: Expectations of behaviour at home may be very different to what is expected at school, in particular explaining yourself, or challenging others, may be unacceptable at home; it is also possible that the nature of knowledge and the notion of truth may differ between home and school.
- Social and emotional difficulties: They may have had problems learning how to behave in classrooms, or may have not found a way to get on with particular teachers; they may lack the social skills required to get positive attention; they may belong to a particular group which is expected to have low achievement.
- Lack of specialised teaching: They may have had teachers whose mathematics was not very strong, and who could not find ways to help them learn.

[1] Level 4 of the England and Wales National curriculum for mathematics is seen by some as the ideal minimum level of attainment on entry to secondary education.

"I really love to teach bottom sets, because that's where all the people go who learn in different ways from the teachers, and it is really exciting working with them."

- Limited teaching methods: They may have been expected to learn only in one way, e.g. verbal, when they may respond better visual or physical methods.
- Low expectations: Early weakness in the subject may have consigned them to lower sets where the curriculum did not expect much from them.
- Learned helplessness: This condition is reported as a result of being supported and helped too much and not having developed the self-confidence to do anything without support.
- Reading and writing difficulties: These may lead to a reluctance to do written work, which may lead to underestimating what they can do mentally.
- Language difficulties: Even small language problems, such as not distinguishing between prepositions, can create obstacles in mathematics particularly if previous teaching has been procedural.
- Physical and physiological problems: They may have diagnosed or undiagnosed physical problems, such as deafness, poor sight, allergies which provoke excessive energy or tiredness, sleep problems.
- Cognitive problems: some have general cognitive problems which affect all their learning, such as short-term memory deficiency, and not been shown ways to overcome these.

Here are all the names[2] of our Project teachers:

David Askew

Rebecca Freeman-Moody

Andrea Pitt

Siobhan Thomas

Anthony Broadley

Sara Howes

Andrea Rigby

Claire Fletcher

Linda Juul

Kevin Slater

The teachers were recruited in a variety of ways. Some joined as a result of letters to all the schools in a county, others through ITT partnership connections, or by personal contact. All we asked was that a teacher should be able to commit herself to teaching low-attaining students by focusing on the development of mathematical thinking alongside covering content. Ten teachers joined: 8 experienced mathematics teachers and 2 in their first three years of teaching. 2 experienced teachers asked to join about a term after the start of the Project; 2 left the Project early due to changes of job, and were replaced by one junior colleague and one experienced teacher. They all taught at least one new-to-them class containing a significant number of lower attaining students in lower secondary school - the focus class for the Project.

They knew it would be hard to persuade the students to respond to mathematics in a positive way. There were a few students who could not be persuaded to engage with mathematics even after two terms of effort, but the majority of students eventually started to do better. 'Doing better' means different things to different teachers, but the following criteria were shared:

- Being more active in lessons, for example by participating in discussion, asking and answering questions, volunteering for tasks, offering their own methods.
- Being more willing to share ideas with others: teachers, peers, whole class.
- Showing more interest in mathematics, for example by doing more homework, working on extended tasks, commenting positively in evaluation tasks.
- Being more willing and able to tackle routine, non-routine and unfamiliar tasks.
- Working more mathematically, expecting to find coherence in tasks, expecting mathematics to make sense.
- Doing better than expected, or than comparison groups, on certain types of question in national and in-house tests.
- Showing improvements in behaviour and attendance.

This booklet is about similarities and differences in the ways the teachers worked.

[2] Sometimes we use actual names in this booklet, sometimes pseudonoyms depending on the context and agreements we have with the teachers.

The main drive for the teachers was the shared belief that all students can learn mathematics, so that the efforts made to help them do so were going to be worthwhile. Even these least-achieving students are entitled to the full breadth of mathematics, including abstract reasoning. The teachers all have similar sets of goals and beliefs in their teaching of students who, on entry to secondary school, have achieved less than target levels in mathematics.

Development of reasoning and thinking

All students are entitled to learn mathematics in ways which develop thinking and confidence in problem-solving. Mathematics offers, as well as numeracy skills, logical reasoning, discussion and argument about abstract ideas, and an arena for careful analysis, categorisation, generalisation. It also offers experience in solving unfamiliar problems, perhaps by looking for familiar structures and bringing knowledge and thought to bear on them.

Right to access mathematics

While many adolescents appear to have obstacles to engagement in mathematics, all students have the right to, and are capable of, full engagement with the subject. Outstanding features of mathematics which make it interesting, and which make learning easier, are the inter-connections between different topics and representations, and the relationships between and within mathematical structures.

Rights and responsibilities as citizens

All students are entitled to have access to the mathematics necessary to function in society, beyond minimal functioning. They need to be able to solve mathematical problems, and solve problems mathematically, with an awareness of number, space and probability, if they are to be good employees and citizens.

Maths as a source of self-esteem

For a variety of reasons, success in mathematics can be a source of self-esteem for students. Some may do better in mathematics than in other subjects, particularly if literacy is also weak. The feeling that they are progressing in a high-status subject can be valuable, particularly if they failed to make progress in their previous schools. But more than this, a positive experience of mathematics can empower them mentally because their own thoughts are being valued. Self-esteem can be developed through:

- Believing that students desire to learn.
- Responding to, using and generating students' own questions.
- Fostering awareness of learning, such as focusing the use of practice exercises on self-assessment.
- Offering challenge and support instead of simplifying the work.
- Enabling students to step out of their comfort zones and take risks, such as creating their own hypotheses (for some of these students, no part of a mathematics lesson is a comfort zone).
- Developing 'togetherness' in classrooms when working on mathematics.

Learners' identity

Learners can see their goal as to learn, or to finish tasks, or to fit in. Some cannot see how to fit in so appear to choose not to fit in. Some see silence and inactivity as safe ways to fit in. In this Project the goal of teachers was to help learners have the goal to learn. Intellectual engagement would be its own reward. There would be no need to construct artificial 'real-life' contexts as motivational devices, although authentic contexts could, of course, be used mathematically where obviously appropriate and relevant for the students.

Taking account of reality

While improvement of mathematical thinking and self-esteem are appropriate goals for educationists, students also need to be achieving in ways recognised by the outside world. So final examination results are important, but students will be better prepared for these if they understand some mathematics, and feel confident to tackle unfamiliar problems, while being sure they have some fundamental tools, such as arithmetic and calculator use.

In this section we report on aspects of practice which the Project teachers and researchers saw as significant in their work. For example, teachers may have changed some feature of the way they taught as a result of thinking through their practice with others, or may have become aware that something about the way they taught was different to how colleagues or other observed teachers worked.

You may not be surprised by much of what we write, because it is largely a description of possible details of good teaching. Why devote this booklet to it then? Because the expectations these teachers had were hard for these students to meet, for whatever reason. We assume that the resistance existed because these expectations were unusual for them; they were out of the habit (or had never established the habit) of working effortfully and successfully in mathematics lessons.We have collected the characteristic features of their teaching under these headings, which are arranged in a continuum. This continuum describes how teachers create an atmosphere in which students gradually become active learners of mathematics, then the nature of that activity, and then the tasks and structures within which this took place.

- Establishing working habits
- Generating concentration and participation
- Interacting and responding verbally
- Reacting to correct and incorrect answers
- Giving time to think and learn
- Working on memory
- Visualising
- Students' writing
- Students' awareness of progress
- Giving choice
- Being explicit about connections and differences
- Dealing with mathematical complexity
- Typical task types
- Structures of lessons
- Extending tasks
- Differences in practice for different classes?

The importance of the role of the teacher was very evident in these classrooms: they were active and purposeful. At any given moment in the lesson, an observer (and presumably the students) could easily see what the teacher was doing and would understand, either because it had been made explicit, or because it was an obvious response to the needs of learners, why it was being done. Thus the goals and purposes of teaching and tasks (which were usually about learning and very rarely about upcoming tests) were known to students.

We do not pretend that these lessons were always good, always smooth, always rosy. They were not. For example, in many lessons Simon would deliver a constant loud monologue of this kind:

> *"It is boring; I can't do this;*
> *I can but I don't want to;*
> *I want to go home;*
> *she knows we can do this*
> *already."*

In one observed lesson, the first of the day, Mandy started with

> *"I haven't got a maths book, Miss,*
> *I ripped it up and put it in the bin."*

and proceeded to take twenty minutes to calm down, demanding attention frequently. She said that already that morning she had fought with her mum, left the house without washing, been spat at, and had a headache.

These really are students who have the potential to be difficult, alienated and disruptive in any school. But the teachers all persisted with their high expectations and what we are reporting is typical of their aims, purposes, goals, lessons and, in general, what was achieved.

In the pages which follow we tell you about how individual teachers acted to achieve similar ends.

Teachers used a variety of ways to develop helpful learning habits. Some worked explicitly on contradicting ambient assumptions about behaviour. Statements like *'they can't concentrate'* were taken to mean *'they don't concentrate'; 'they can't listen'* as *'they don't listen'*. Teachers would then try to create lessons in which *'they do concentrate'* and *'they do listen'*. All of these took time, creating new habits takes new routines which need constant repetition to become habitual. The teachers did not give up; they kept plugging away at creating new habits. Here are some examples of what individual teachers did:

- To establish listening, aural questions were used at the start of lessons as a settling-down task.
- To establish effort, students were made to look at a problem on the board in silence, told that the aim was not for them to solve it, but to think about it, and then they were asked for ideas. Over time, as this became routine, more and more students actually did think about the given problem.
- To establish atmosphere, effort, concentration and memory, students in one class were given the same sheet of multiplication facts to complete at the start of every lesson and worked through it in silence for 15 minutes. The aim was to complete more and more each time, to compete with previous personal best.
- To establish listening and concentration, answers are read out once only, very quietly.
- To establish the expectation that everyone will be thinking, students know that anyone in the class might be asked to answer, after some thinking time.
- To establish the expectation that everyone will be talking about the maths, pairs of students discuss maths and all pairs have to report back on what they have found.
- To establish homework organisation habits, students had to first establish the habit of taking paper to and fro from school successfully! When this was established, they had to write something they had learnt on the paper.

- To establish discussion, students were seated in a boardroom / horseshoe style so they could see each other to talk to. Although they could still sometimes use this as an opportunity to be silly, the teacher used this style often and students responded well more and more frequently.
- To establish real discussion, so that the teacher is not the centre of all interactions, one teacher stopped sitting at the 'front' of the class and sat at the 'back'.
- To establish continuity of work, one teacher keeps every student's work in their own plastic bag, so that at the start of a lesson the student has something to open and start doing, containing tasks and artefacts which trigger memory from the last lesson.
- To establish the recognition that the kind of answers which, though correct, students had been giving for years (and been praised for) were no longer appropriate, one teacher would ask for: *'my favourite answers?'* and praise with *'that is one of my favourite answers'* to indicate that it was time to move on to harder things, and notice when someone did so.
- To establish the need for a growing storehouse of knowledge and techniques, students were given special books in which they would write their chosen examples as a reminder.
- One teacher required sensible, thoughtful, participation in public mental arithmetic work, using the rest of the class as collaborators, and managed to develop a no-risk atmosphere in which this was achieved.

Behaviour can be altered, but it takes time, persistence and imaginative methods. Old habits have to be replaced by new ones over time. A 'training' approach(with clear expectations and rewards) might be effective. Anything which disrupts old expectations (including expectations of the teacher's behaviour) is worth trying. Time has to be given in lessons to establishing new habits, and time must be given over several weeks for them to become 'habits'.

Here are some ways in which Project teachers explicitly generate a studious atmosphere of concentration and participation. Of course this takes time, but in this Project no one gave up trying to achieve these. We noticed that participation does not always imply concentration. One can be involved physically, such as taking part in a physical representation of a graph or of a data set, without concentrating on meaning. However, to concentrate one has to participate to some extent.

Teachers worked on concentration and participation by:

- Structuring whole class tasks so everyone has to take part.
- Asking students to check and make comments about each other's work.
- Making *'listen to each other'* important by asking for students to say it in their own words, or to take each other's ideas into account in their answers, or to do some work based on another student's idea. This could be done implicitly; for example, one teacher asked:

  'Can you say that again to Mark because you are saying it very well?'
- Getting students to listen to, or read, questions in detail before they answer.
- Giving students the responsibility to monitor their own efforts. Again, this can be done implicitly, emphasising high expectations in apparently casual interactions.

  Teacher: *Liane, are you working well?*

  Liane: *No*

  Teacher: *Are you doing something about it?*

  Liane: *Yes, miss, I will make it good work.*
- Asking students to imagine something and giving time for an image to develop.
- Using visual and kinaesthetic teaching methods as well as verbal and practical methods.
- Using rewards such as chocolate used to start participation, until a sense of intrinsic interest has a chance to develop (so that chocolate becomes irrelevant).
- Recognising that some students have difficulty doing homework for a variety of reasons: the task may have been posed in an ambiguous or vague manner, it may really be too hard, there may be problems at home and so on. Flexibility about homework expectations was the norm on the Project; some ran homework clubs, or used school detention sessions as a maths help clinic.
- Memory was explicitly worked on by some teachers; sometimes they would say *'you need to remember this so listen hard'*, others would ask students *'how could you remember this?'* or *'how did you remember this?'*
- Each student may have to produce something to contribute to the class; these can be compared, classified, or ranked according to difficulty.
- Students are asked by all teachers to explain ideas, concepts and definitions in their own words.
- Some teachers deliberately ask questions with multiple answers which everyone can answer to ensure all can participate, and to generate enough answers for the class to consider.
- All teachers use *'hands-down think'*, where no one is expected to raise hands to answer. The class all sit and think and anyone might be asked to answer. There is extended time to think about their answer.
- Students are asked to give answers to worked questions, rather than the teacher or textbook giving them.
- Students create questions for each other.
- Small whiteboards are used so that everyone can show an answer or idea or example.
- Teachers explicitly use what is interesting in students' work to direct the next part of a lesson: *'Can we have a look at something I have just noticed in Sam's work?'*
- One teacher gave time for students to cue in to the mathematics by starting with a silent thinking task which related to what had been done in a previous lesson.
- One teacher uses elaborate stories with a beginning, middle and end to exemplify

mathematics in context; thus she hopes her students will build a complex picture in their minds of the story, the situation and the mathematics which they can relate to personally and emotionally.

There is a connection between engagement and learning; students cannot learn unless they are engaged, and engagement is a combination of social, emotional, intellectual and task characteristics. Teachers had to work on all these facets to ensure engagement. All teachers believed that learners' concentration and participation could be developed.

Interacting and responding verbally

One of the most exciting things to emerge from this Project is the variety of ways in which interaction in classrooms between student and student, or between teacher and student, promote learning. It is particularly noticeable that most students learnt (or relearnt) to talk to each other in useful ways when that was the clear expectation of the teacher. This is in contrast to the common habit with such groups of separating students and expecting them not to interact. Teachers would vary the room and seating arrangements, for example putting tables in a horseshoe, or placing themselves near the back of the room, in order to 'upset' normal patterns of interaction and promote more student-student talk. We are not pretending that all interactions were about mathematics, but instead we identify the kinds of interactions which were about learning and give examples of these. To over-control interactions means that these would also be lost as well as the distracting kind, but this meant that improvement did not happen overnight, nor was improvement sustained in every lesson.

Student-student interactions

- Students listen to replies from other students to questions asked by teacher.
- Students are invited to write on board, or on OHP, so their ideas can be discussed.
- Students generate examples and methods, which are shared publicly and students make comments about them voluntarily.
- Students working together manage to coerce reluctant peers into the activity, creating peer-pressure to work. One student said to another, who had just interrupted the teacher: *'just be quiet, then you'll be alright, won't you?'*
- Students in one class are free to move around and discuss their work with others; one is observed to tell another she cannot do it and they work together for a while.
- At times, all utterances are directed towards the teacher; at other times, students talk directly to each other, even in whole class discussion.

- Students mark each other's work and discuss dubious answers.
- Tasks which clearly require discussion are sometimes given, e.g, those where more than one approach is possible, or where ways to get started are not immediately obvious.

Student-teacher interactions

- Teachers have standard questions which they use frequently enough that students expect them and can ask them for themselves, for example *'what is the same and what is different?',* and *'is it always, sometimes or never true that ....?'*
- Teacher asks questions, which might be open, closed, or structured in ways which gradually expose a topic, or an idea, or a generality of mathematics.
- Extended waiting time is used, before and after answers are given.
- Students give answers and methods or reasons.
- Students might be invited to comment on each other's answers - the teacher may not indicate which are correct or incorrect, or which can be further developed.
- Teacher expects students to use technical language and asks for such uses to be explicit and repeated.
- Discussion sessions can consist of remarks all channelled through the teacher but are more likely to involve interchange between many, orchestrated by teacher.
- Teacher writes everything students say on board, for later discussion and criticism by whole class.
- Teacher invites students to share in exposition, or in worked examples, not just by 'gap-filling' but by offering suggestions and reasons, or saying how one line follows from the one before.
- Students ask questions about what they do not understand, or about other possibilities.

- Students ask for, and are given, individual help: teacher may demonstrate how to do something, or give instructions, but is more likely to probe what the student thinks already, or to ascertain what they understand.
- Teacher offers counter-examples to students' conjectures to promote further thought.
- Teachers see their role as helping with learning, not mainly as markers of work or keepers of the peace.
- Some students need help from the teacher in developing organisation strategies in order to learn.

The quality and patterns of interaction can be changed so that students participate in the creation of mathematics in the classroom. Gradual shifts have been made so that students are more responsible for their own learning, and what they do is valued by the teacher and other students.

One particular type of interaction which is worth separate attention is the teacher's reaction to answers. A teacher can choose to react to all answers, right answers, wrong answers, or to not react at all. These decisions create different dynamics in classrooms, and sometimes not reacting at all promotes further thought more effectively than comments from the teacher.

Listening to pupils' responses to questions is vital for 'reading' the lesson and deciding where to go next, to move on, to recapitulate, and to provide more rehearsal space. The less articulate the students, the more agile the teacher needs to be.

Students in the Project classes learn that incorrect answers are valuable in providing learning opportunities. It is frequently the case that students who feel free to answer in an unthreatening atmosphere will self-correct as soon as they hear themselves saying something. The Project teachers are good at developing reflection on incorrect answers, and also on encouraging students to think further.

- A student has written an incorrect answer on the board and the teacher says: *'We are going to leave this on the board as, even though it is wrong, it might give you some ideas about what to do'.*
- Some teachers say nothing about correct answers and continue to ask others until several students have also given correct answers.
- Some teachers ask for explanations of both correct and incorrect answers (thus giving students a chance to self-check and self-correct).
- When students cannot answer, teachers avoid simplifying the questions until it becomes trivially answerable. One teacher asks *'how else could I ask this?'*; another coaxes, saying things like *'you remember when we did such-and-such and Simon said so-and-so?'.*
- Teachers are sensitive about the inability to answer questions, and recognise the social need for some to look as if they are participating, or to look as if they are not. Stewart had his hand up and was chosen to answer by the teacher, but said: *'I have forgotten',* the teacher just said: *'Alright, you've forgotten'* without making an issue of it.
- It is very rare for Project teachers to accept the first correct answer given without further opportunity for learners to think about it. For example: *'you answered that much too quickly; what other ways are there of working out a quarter?'*

In general teachers:

- Provide a range of cues for the maths.
- Are interested in what students know and what they do not know.

  Tim: *I don't know nothing, Miss.*

  Teacher: *Let's go back to what you do know then.*
- Develop expectations of interest, e.g. by saying: *'Whatever you know I am interested in knowing'.*
- Expect all to be involved and to have some thoughts about the questions asked.
- Shift the responsibility for *'knowing what to do and how to do it'* to the pupil.
- Use their responses to pace the lesson: *'Do we all agree with that?'* when it is time to move on; *'Tell me more'* when reflection is needed.

Answers, whether right or wrong, are material for consideration and for learning, not end points except where speed, accuracy and memory are the focus of the task. Answers, therefore, are often given the status of conjecture to be tested out and discussed.

Rather than rushing through topics, the teachers gave extended time for learning, and often discussed this with students explicitly. They believed that it takes time to reach a learning goal, or make a new connection, although the learning which is going on *en route* may be dense and busy. Discussion takes time, and students would benefit from being aware of the use of time to explore, mull, think again, to ponder and so on.

There were various ways this expressed itself in practice:

- Some teachers extended a single topic over several weeks, in order to use many different representations, to ensure progression in the topic, and to include repetition, use and exploration.

Deep progress was ensured by such extended tasks, and deep progress also included the development of work habits which allowed long tasks to be successful. Teachers gradually shifted learners' perceptions of how to pace their working.

- One teacher put no time limits on any task, and explained this to students: *'sometimes you need more time to think'.*
- Ideas might be shared before anyone felt they had finished, so that everyone might be helped by hearing how others were working.
- Students can only learn at their own pace, not at an externally imposed rhythm[3]. This is in contrast to completing a procedural task within a given time, which may not result in learning directly but may provide raw material for later reflection.
- Exercises are provided which need not be finished, but which offer choice and challenge as well as practice. If the expectation is that students will learn as much as they can from an exercise, rather than merely 'do' as much as they can, then time taken to finish is not an issue. One teacher would say: *'Take your time, go through very carefully ... there is no hurry'.*
- Students can ask for more time to work on an exploration.
- Students who gave wrong answers to mental calculation questions were told by one teacher *'you answered that much too quickly',* thus indicating the value of self-checking against speed.
- Class discussions were often managed so that there was no apparent time pressure; for example, there was time to consider statements carefully, time to find out what everyone knew.
- Sometimes there was implicit pressure because other students would be waiting for someone to answer or finish something; other times there was time pressure used constructively, such as in chanting to a rhythm in order to help students attain fluency aspects of mathematics.

In general, teachers thought about the appropriateness of timing from the dominant point of view of learning, not from a dominant point of view of coverage. Rote-learning was avoided, although some aspects of mental arithmetic benefit from chanting and memory. There was a balance between time pressure used to encourage fluency and effort, and space used for thinking, reasoning, considering, reflecting.

Covering the whole curriculum was important, but so was understanding what had been learnt. Working towards understanding is ultimately more long-lasting than mere acquisition of procedures.

[3] 'Own pace' can be slowed down by disengagement, but here the focus is on engagement so that working slowly does not equate with 'wasting time'.

One of the roles of the teacher is to help students know what is worth remembering in mathematics. Students need to know what it is valuable to remember, what will be useful in future if it is fluent, and how to remember these. *Overloading* struggling students with meaningless mnemonics was not seen as helpful, although having some of these around can help students feel equipped for mathematics. The Project teachers focused instead on memory with meaning, sometimes trying to link words with visual or physical memory. Memory was worked on at several levels: remembering from lesson to lesson provides continuity and a sense of short-term progress; working deeply on mathematics can aid longer-term memory, and memory for mathematics can aid deep progress, and contribute to self-esteem when students are aware of the extent of what they know. Hence teachers use a variety of ways to develop memory and here are some examples:

- Reviewing areas of confusion through discussion in pairs of distinctions between, for example, area and perimeter, or mean, median and mode.
- Reviewing at start of lessons what was done in previous lessons. One teacher does this by giving a related task for silent consideration to start the lesson.
- Ensuring new topics are mathematically linked to previous topics, not fragmenting the subject.
- Reminding students about central aspects of topic frequently, such as technical terms, definitions etc., as if they are tools for the task. For example: *'Maybe someone can run through what 'mean' means for us.'*
- Pointing out what they might need to recall, saying *'listen carefully to this'.*
- Using pooled memory of class to create concept map, or spider diagram on board. For example, one teacher wrote *'Facts'* on the board and asked: *'Who can tell me anything about polygons?'* She then wrote everything which was said underneath it, to be tidied up later.
- Whole class discussions used to recreate techniques.
- Implicit and explicit work on calculating and holding information: *'you have to sort this out in your heads; it'll help you remember it'.*
- Getting students to return to a physical position, or location in the classroom, which related to some previous activity.
- Asking them to devise their own methods of recalling, and of testing their recall.
- Asking them how they remembered something.
- Repeating something which would be useful to remember enough times for it to become familiar[4].
- Combining words and movements, such as when writing algebraic expressions *(drawing the fraction dividing line while saying 'divided by')* or when learning multiplication facts *(sequence of actions accompany tables)*, or when drawing angles *(indicate turn with hands as they say 'angle').*
- Having objects, equipment, which is used frequently during work on a topic so that handling the equipment brings back the memory of what it was used for.
- Using writing to develop memory for chains of reasoning.
- Using mental mathematics to help develop memory and thinking skills because they have to sort out the mathematics in their heads while 'holding on' to numbers and relationships.
- As we have said before, memory was explicitly worked on; sometimes teachers would say *'you need to remember this so listen hard'*, others would ask students *'how could you remember this?'* or *'how did you remember this?'*

Memory for what is important needs explicit work; this includes frequent return to main ideas, so that memory is necessary and important. Students need to know what is worth remembering. Explicit work is needed on words, techniques, facts and images to help conceptual recall. For most students, memory can develop, particularly if there is some self-esteem to be gained by doing so - so remembering mathematics needs to have value in the classroom and to be used as a marker of personal progress.

[4] *This is only used selectively for fluency, not as a usual form of teaching techniques.*

All Project teachers emphasised visualisation for several reasons: many of their students were better at visualising than verbalising and images may be easier to recall for some students. The spoken and written word are sometimes insufficient for pupils to make meaning, so alternative forms of representation, or indeed many forms of representation extend the possibilities for learning. Sometimes visualisation was done mentally, sometimes through drawings and diagrams. Visualisation appeared as a feature of practice in these ways:

- Explicit appeal to the imagination in spatial and in algebraic tasks *(e.g. using an empty bag to represent an unknown quantity).*
- Students' own images are compared to another image (given verbally or graphically).
- Working in the imagination, then checking some other way. For example, Kevin asked students to draw reflections and rotations by imagining what might happen, then drawing it. He then gave them tracing paper to check their work. This emphasised the transformation process. If they had used tracing paper first, they may have concentrated on the detail of producing the image.
- Making the contents of a 'toolkit' of visual and tactile objects with which to pursue several tasks. This also gives concrete ownership; handling the 'kit' gives instant recall of past tasks.
- Appeal to numberline or other spatial representations of number while doing calculations, *(e.g. a quarter is half and half again; 1/3 x 3/4 visualised as one third of three parts of something).*
- Focus on mental calculations.
- Spider diagrams and concept maps are used to convey links in mathematics, or as tool for awareness of learning, such as through self-assessment.
- Use of equipment. For example, students made multilink towers to create physical data sets which would have particular properties: means, modes, ranges etc.
- Use of physical actions to represent mathematics *(e.g. walking to and fro along the numberline for negative numbers; making graphs by standing in appropriate places on a floor-grid etc.).*
- Asking students to re-enact previous visualisations and actions.

Students can use a range of senses to engage with, understand and remember mathematics. Teachers can offer a wide range of approaches, including asking students to imagine visual, tactile and physical experiences, both recalled and totally imaginary.

The use of visualisation may be planned thoughtfully and deliberately, comparing what is likely to be learned using different forms.

Teachers were very aware that writing may be a problem for their students, but that writing is a valuable means of communication in mathematics. They thought carefully about their use of writing and their expectations of students. They provided models, and scaffolding, for writing, but also minimised its use where it might be a barrier to learning mathematics. Here are some examples:

- Use of writing frames so that students fill in blank spaces rather than have to write arguments in full. Some students were asked to fill in this sentence: *'when I looked back over my answers I saw that these were similar . . . , and these were different . . . '.*
- Students are encouraged to make their own notes about what they notice, or what they would like to investigate next.
- Teachers write students' ideas on the board in their own words (or students write their own ideas on the board). These can be hypotheses about concepts and tasks, calculations, student-generated definitions, etc. Similarities and differences are discussed.
- Students draw sketches or diagrams of spatial work they have done tactile objects such as cut-outs, tiles, etc.
- Minimal use is made of copying teachers' words, or producing written work solely to satisfy external audiences such as parents and inspectors.
- Mental work is emphasised where possible.
- Use of colour where this aids thinking[5], shows classification, or contributes to self-esteem through ownership. One student said: *'I like colouring-in, it helps me think'.*

The relationship between writing, thinking, learning and remembering is neither obvious nor trivial, especially for students who may have difficulties with general literacy.

Teachers can consider the role of writing and weigh the obstacles against the benefits, providing support, frames and scribing where this allows students to focus more on mathematics.

The effort of deciding what to write about mathematics can scaffold and support the development of conceptual understanding and reasoning. It can also support students' certainty about concepts, and their memory for chains of reasoning.

[5] But all teachers were very wary of allowing students to 'colour in' where it was not obviously connected to thinking about mathematics.

Students in lower sets often find it hard to know when they are making progress in mathematics. Self-awareness is very important to their self-esteem, and that there are a variety of ways in which this can be established in mathematics lessons, from the casual to the more formal. Teachers expected students to engage in a wide range of assessment activities, including self- assessment tasks, traditional testing, and so on[6]. Self-assessment usually involves reporting to the teacher, while teacher-assessment, however informal, involves reporting to the student.

In this section we focus on something more subtle. The emphasis in this section is on the learners sensing that they are making progress throughout their work because the way the task is structured offers clear markers of progress. Here we list ways in which students in Project classes became aware of their own progress:

- Students' casual discussion about their progress during tasks is a source of self-monitoring relative to others. Even tiny interactive incidents can make a difference:

  Mhairi (calling across the room): *June, we've done it!*

  June: *so have I!*

- Sharing ideas, especially during tasks, can give a sense of progress relative to others and the task.
- Use of a range of ways of recording work on paper, formally and informally, in words or diagrams, can give a sense of how much has been done, although not all the mathematics done can be usefully recorded.
- The teacher giving time to listen to students' ideas seriously, or recording all ideas on the board, can give a sense of achievement.
- Teachers put strategies in place which show that students are being heard, as well as giving the teacher information.

  Jilly said: *I like it when we have to draw a traffic light next to our work to show how hard or easy we thought it was. I don't have to say how I feel about it out loud, that's embarrassing. This works because the teacher can go over the work again and again if she wants to.*

- Some games-type approaches to mathematics have inbuilt mechanisms for monitoring progress, such as having to remain sitting or standing until some task has been completed.
- Being asked to make up their own hard examples, or to *'make up one you can do now, but you could not have done a week ago',* can give a sense of progress.
- Some students are given a pre-test at the start of a topic so that what they already know can be valued and used, then students are in a better position to identify their own new learning.
- Starting a topic by being given a relatively hard task and students finding that, after a few lessons, they can do it effectively.
- Asking students to write their ideas on sticky labels and stick on a poster and then to refer to them in later lessons as they explore; this helps them to recognise their progress.
- Students are asked to write down their own definitions, methods, reminders when they think they have reached an understanding of something.

The teacher does not have to be the audience for assessment. Students can get an implicit sense of their progress, and hence develop self-esteem, if tasks are structured to include marking progress of some kind at various stages.

[6] At the time of the Project, teachers were beginning to adopt some of the methods suggested in **Black et al** (2002) but this list includes additional ideas

Students were given choice in many ways during the Project, and some teachers were aware they were giving more choice as the Project progressed. This allowed students to become emotionally attached to their work and develop their own ideas.

> *It's boring to be told what to do. It's nice to have time to choose ... kind of relaxes me. When you are told what to do you don't want to do the work ... when you are relaxed you want to sit down and do the work.*

Students often had to account for the choices they made, and this encouraged reasoning, explanation and justification and ensured that choices were made with mathematical understanding.

Opportunities for choice included:

- Choosing their own methods of working
- Choosing their own methods of calculation.
- Choosing a starting example for a mathematical task, such as: *'draw your own quadrilateral, any four-sided shape you like; now join the mid-points of its sides'*; or *'imagine you are cleaning windows of a sky-scraper, choose which floor you are on, now go up two floors, down five etc. etc.'*
- Choosing their own examples to practise techniques. Having worked on percentages students were asked to find as many different percentages of a given amount as they felt able to do.
- Choosing how many examples to do in order to feel confident. One teacher says: *'do as many as you need to in order to feel you know how to do them'*.
- Choosing how hard to 'push' an idea, or extend a variable, to see what is possible.
- Choosing their own variation of a task within some mathematical constraints. Students created their own shapes for working with transformations with vertices on the 'dots' of dotty squared paper.
- Deciding what they want to investigate about a given situation. After discussing results obtained through exploring possible perimeters of shapes made by abutting four congruent squares, students can pose their own questions to explore further questions about squares and perimeters.
- Offering their own interpretation of a mathematical statement. When asked to *'find the sum of three consecutive numbers'*, students restated the problem in their own words. All interpretations were written on the board and compared until agreement was reached about what it meant.
- Teachers might plan lessons so that students can do 'this' or 'that'; they do not have the choice to do 'this' or 'nothing'.
- Students can choose whether to spend further time on a task.

Teachers expected students to take responsibility and provided situations in which choices had to be made to promote this. They expect as much self-direction from students with lower achievement as from any other group.

Teachers draw students' attention to features of mathematics which are of a more general nature than the current topic. The teachers do not expect students to know what is worth noticing, so they incorporate comments about good mathematical habits as they arise. They are sometimes very explicit about features, ways of thinking, and relationships which a stronger student might notice automatically. Some of the following examples are about learning to make connections in their work and ways of working; some are about learning to discern subtle differences and features of mathematics. Some of them may seem obvious, but being explicit about some of the connections and distinctions we take for granted can be very useful for students who have previously not been making the connections their teachers expected. These processes can be applied within or between mathematical tasks, topics and experiences.

Within tasks:

- Students check validity of statements for themselves, so that they learn how statements and conjectures relate to examples they invent.
- Using a particular layout of a calculation on the board, the teacher changes the numbers being used and says: *'the information I am giving you has changed'*, thus indicating that students ought to pay attention to differences.
- Students are asked: *what is changing and what is staying the same?*
- When working with the perimeter of shapes made up of four abutting squares, the teacher asked them to consider the effect of each arrangement on the perimeter, thus getting them to connect variables, and also inducing a pre-conjecture state of understanding that the variables might be related.
- Asking students to articulate different methods for finding one answer, a common practice, the teacher then gets them to evaluate and discuss the power of each method.

Between tasks:

- When multiplying positive and negative integers, explicit connections were made to students' experience with natural numbers.
- When discussing the symmetries of a cube, students are reminded about their recent experience with dice and the related probabilities of getting scores, thus linking both to the shape of the cube and showing the value of such links.
- When working with fractions, one teacher made connections with areas of rectangles which were not trivial, i.e. which encouraged more thought about both.
- When students found themselves stuck, the teacher asked: *'how did you get out of a similar situation before?'* thus indicating they could make links based on similarities in the situations, and working on their resourcefulness by modelling how to deal with being stuck.
- Students are asked to make their own distinctions between mathematical objects which seem superficially similar, such as: means, medians and modes; bar charts and histograms; expressions and equations; equalities and inequalities.

As well as giving students the opportunity to notice changes, links and similarities in mathematics, teachers can make these moments explicit. Students learn more about what is expected, what would help them understand more, and how to look more searchingly at mathematics in future.

Project teachers did not simplify mathematics for students, nor did they fudge difficult issues. Instead they saw their job as helping students learn mathematics, with all its complexities. As well as offering ways to deal with complexity in mathematics, each of the tactics which follow is also an example of learning to think in appropriate ways for mathematics:

- When working algebraically, students were encouraged to use a wide range of letters, not just the usual ones, to signify unknowns.
- All teachers discussed similarities and differences between mathematical objects and examples, thus modelling the processes of classifying which would help students make sense of variety in mathematics.
- Typical points of confusion, such as area and perimeter, were tackled head on, with exploration of shapes which increase area while reducing perimeter, and vice versa.
- Suitable levels of accuracy were discussed and students given choice about which to use and when.

> **Students had been working on** *'squeezing a number between 64 and 88, between 0.007 and 0.009, between 7.09 and 7.11'.*
>
> **Teacher:** *A lot of you would like to have your answers checked.*
>
> **Students, shouting:** *Me, me, me*
>
> **Teacher:** *Shall I read out the answers?*
>
> *[pause]*
>
> *How many answers are there?*
>
> **Students:** *Millions*
>
> **Teacher:** *So can I read them out?*

- Proper terminology was generally used, with students given time to learn how to use it.
- Students were asked to consider whether statements were always, sometimes or never true.
- Students were encouraged to give detailed answers about what they saw in diagrams or algebraic expressions.

Students can learn rigorous mathematics if the relevant kinds of rigour are discussed explicitly. Students can learn complex mathematics if there is discussion and other forms of verbalisation to help them sort out the complexity.

The aims of all these tasks are complex. The mathematics is not simplified; there is no sense of 'finish' in the tasks, since they are stated in ways which require extended thought; there are supporting props in place (chocolate, materials, discussion); they all involve reasoning of some kind; they all contain personal challenge. If these learners react differently to others, it is that it takes them more time and they may need more support because they need to develop resilience, resourcefulness, language competence or memory.

A. Teacher writes numbers on the board: 21, 4, 1, 5, 5, 3 and the aim is to get 21 by using all the numbers. This is to be worked on at home, but is voluntary. Taking part earns a small piece of chocolate, finding a lot of ways to do it earns a larger piece of chocolate. There are some ideas generated in class, mentally, before they move on to other work.

Comments: Chocolate is used as a playful reward. The underlying message is that home work should be done, but that students who have no useful work habits, and no self-esteem, need coaxing back in through creating a no-risk atmosphere. Thus help is generated in the lesson so all can understand what to do. The aim is not to find one method, but to find as many as you can; thus continued, extended exploration is being encouraged.

B. After reminding students about different kinds of average, they are given multilink to build towers. The aim is to build a family of towers which together have given averages, such as a mean of 4, a median of 3, a range of 5 and so on. Families which work are recorded on a worksheet.

Comments: Students are reminded that it is important to recall different averages. The task encourages exploration of how averages are made up, and awareness of the variation possible. It focuses on structure of averages, rather than methods of calculation. The worksheet frames the writing so that recording is not problematic, nor is time taken away from thinking by writing, drawing up tables etc.

C. A new topic is introduced part-way through a lesson: polygons. Students discuss what they know about names and properties of polygons and all these 'facts' are written on the board. They then draw two separate circles using compasses. The homework is to think of ways to cut their circles (pies) into various numbers of equal pieces and to think about 'how can I be sure my circle is cut into equal pieces?'

Comments: The pooled knowledge provides the starting point for new work; through this the teacher assesses what they know and also provides everyone with a place to start thinking. The homework is open-ended and will generate the raw material for the next lesson. The task is posed as 'being sure'; thus it is about reasoning, rather than just doing what is asked. The circle-drawing creates a link between topics, and the potential link is not obvious from this lesson. Rather than make this explicit, the teacher retains an element of surprise. In the next lesson, the sectored circles will be used to generate discussion about properties of regular polygons, but could equally well be used to remind students that fractions have to be equal parts.

D. Students each have a pack containing 10 cut-out congruent squares which they made in a previous lesson. All students have explored how to get maximum and minimum possible perimeters when four squares are arranged edge to edge in various configurations. Results are discussed. They are then asked to create their own questions as directions for future explorations. These ideas are shared so they can abandon their own question and choose another if they like.

Comments: The reasoning processes they have experienced with the 4 squares, supported by whole class discussion, can be re-used in cases with more or fewer squares, thus they will get a sense of progress as the reasoning will be easier second time around. They can monitor their own progress by knowing if they have answered their next question or not. The frequent use of the word ‘perimeter’ is important, and varying the number of squares allows variation in area. The availability of more squares encourages this choice. The personal packs create ‘ownership’, familiarity and memory.

E. Students are presented with pyramids, in which they are introduced to a structure which has the rule ‘add the contents of two adjacent cells to get the content of the cell below’.

In this class the task is varied so that:

22	7	13
29	20	
49		

(a) they have to fill in missing cells from lower down the pyramid, given the upper cells.

(b) some cells have algebraic entries.

They are then given a work sheet to do, the final challenge being to make up some of their own.

Comments: This familiar task[7] has not been simplified for lower attaining students. They get a complex version which requires symbolic manipulation, reasoning, ‘working backwards’ from answer to question, and also have to reflect on what they have done by creating their own questions. Thus they have been treated as thinking beings who can deal with complexity.

[7] See ***Journey into Maths*** by Alan Bell, David Rooke and Alan Wigley. Published by Shell Centre (1978 &1979) www.mathshell.com

Structures of lessons

Here we give some examples of lesson structures to show the variety used by Project teachers. There was no uniform lesson style, although all contained some discussion and sharing, not always as a whole group, some mental work and some reflection on previous work. Contrary to a common belief that low attaining students and ‘difficult’ classes need frequent changes of activity, these teachers worked on developing sustained activity and opportunities for thinking about what they are doing.

Lesson A: Aural questions used as a settling device, students work in silence on their own; followed by discussion of answers and methods in which they mark their own work. Then there is an activity which starts with the teacher demonstrating and orchestrating discussion about possible ways to tackle tasks. Students work on a worksheet which includes being asked to make up their own questions to tackle. The ‘own questions’ tasks act as a reflective device; there is no final whole class time.

Lesson B: Students continue work started in a previous lesson, writing down their findings on small whiteboards to share; they are finding ‘odd ones out’ from a collection of shapes, according to some constraints. After this there is discussion about what was found in previous lessons, and they create their own questions to continue exploring. At the end, students share findings and use OHP to show what they have found.

Lesson C: Students start with a number challenge; students volunteer to write their suggestions on the board which have been worked out mentally. About half way through the lesson the new topic is started: data-handling. They brainstorm everything they recall about it. They have to each invent five questions to ask people and the whole class discusses potential difficulties with collecting data to answer the questions. Almost the whole lesson is whole class discussion.

Lesson D: Students are given a worksheet and materials to cut out and manipulate to find reflections without tracing paper or mirrors. They make up two questions to be answered by their partner. Their work is checked by constructing the ‘same’ reflections using tracing paper. There is no whole class discussion.

All teachers found themselves working to sustain students' interest in a topic over time, not varying the task and topic frequently, but encouraging deeper thought in a variety of ways. This often worked against normal practice of providing students who found it hard to concentrate with frequent changes and new tasks, or changing topic frequently to avoid boredom. Some of the ways this was achieved were:

- When a student has 'finished', the teacher shows how further questions can be posed about the same problem situation.
- Teachers ask *'why?'* and *'what if?'* to encourage reasoning and justification, or further exploration of changes of different variables. This also models the kinds of deep questioning which students can do for themselves in future.
- When students are asked to create their own examples, they are encouraged to make up really hard ones which might present new challenges to work on. Some students who had been making up their own scalars for enlargement activities were then asked to explore what would happen if they used negative scalars.
- All hypotheses are followed with *'why?'*
- Further changes can be made to a problem situation, allowing students to apply what they have learnt so far. Having found out that when two lines cross you get two pairs of equal angles, which are supplementary, students are then given a third line and asked to find out about the angles you get if two of the lines are parallel.
- Reflection on worked examples which students have completed, asking them to say which were easy and which were hard, and why, makes the whole exercise the focus of study rather than the technique on its own.
- Giving something familiar, in an unfamiliar form, helps make connections as well as doing the task. Some students are given a 100 square, but in a different lay-out. It is in a 'snake' shape, with 100 in the top left corner and 0 in the bottom right hand corner. The teacher's aim is to use students' sense of comfort from using the familiar 100-square to think afresh about number relationships.
- Offering repetition for familiarity, but with significant variations each time, achieves fluency through accumulated experiences and gives students a sense of the range of possibilities in a topic. Some students had completed some work on ratios of two quantities successfully. They were then offered similar work (demonstrations, practical tasks, exercises) on ratios of three quantities. This helped them become more fluent with two, and also gain more understanding about ratio, and then some asked if they could do four!
- *'Learn something new'* and *'learn as much as you can'* are goals for lessons.
- The opportunity for students to make a personal toolkit for a topic, so that opening the toolkit at the start of each lesson triggers memory for what was done before, also provides raw material for new questions. Some students compiled their own list of useful percentages and how to work them out; looking at the list suggested more which could be found, so students developed their own 'harder' questions.
- All situations in which there are various mathematical objects or examples produced, or statements such as facts or conjectures, provided opportunity to work on reasoning and higher levels of abstraction.

Teacher (having written Wayne's idea on the board): *Can I add something or change something to make that statement true?*

Teacher (sorting a pile of statements about quadrilaterals): *Are there any statements which do not work together?*

Teacher (looking at several decimal equivalents of unit fractions): *What is the same and what is different about these?*

Sustaining work on one topic over a period of time promotes deep progress, awareness of progress and hence self-esteem and a sense of being a learner of mathematics. Concentration and participation enable tasks to be extended, through learners being actively engaged in thought. The relationship between these aspects is complex and non-linear; one does not guarantee another. There is no such thing as 'a guaranteed extendable task'. The extension is created by the class, the task, the teacher and the learning.

Differences in practice for different classes?

Some of the teachers used the same methods with all the classes they taught from the start. Others found that methods they developed during the Project with students who had previously lowest attainment worked well with other classes too. At the end, we wondered what was different about teaching the lowest attaining adolescents.

Most teachers claimed they did not teach differently. They had the same expectations in terms of task complexity, effort and behaviour in all their classes. Some teachers used the same tasks with several classes, including higher sets.

Outside observers would say that they *did* teach differently, because they incorporated into their practice deliberate strategies to change habits. This took time. It took students time to realise that they need to adopt new habits, and that they will benefit from them. Teachers and students took time to find out what was already known, and to establish understanding before moving on.

Project teachers saw short-termism operating in general in the education system, typified by:

- Curriculum coverage needing to be completed before the next test;
- Students expected to display good work habits immediately and punished if they do not;
- Moving rapidly from one task to another if concentration wanes;
- Focus on finishing work, either in class or at home.

Teachers and learners in the Project were working to a different timescale. Short-termism had been abandoned and long-termism was put in its place.

- Longer was spent on each topic than was recommended nationally, but content coverage was still important. Coverage without understanding and memory was pointless - time was spent on understanding and memory.
- Longer was spent on establishing good work habits and undoing previously developed habits. If this took a lot of lesson time, so be it.
- Longer was spent on thinking and on particular tasks, to establish participation, reasoning, understanding and connectedness.
- Focus on learning as much as possible, rather than finishing tasks.

What did the teachers need?

They needed to be brave because some of them were working in ways which contradicted normal practices and expectations. They also needed to be brave because the way they ‘covered’ content with their students did not conform to an easily-observed list of topics. Working with others over time, as you can do in a research project or professional development course, helps.

They needed to be clever in :

- Finding and creating time to spend on what matters.
- Finding and creating tasks which:
    - expose what is already known,
    - use this knowledge,
    - use a range of representations,
    - provide opportunities for mental and mathematical challenge,
    - include markers for progress
- Interacting with students in ways which sustain effort, which sustain challenge, and yet which build self-esteem.
- Creating atmosphere and interactions which immerse students and teachers in mathematics.

How did the teachers know their students were 'doing better'? They said:

Working on understanding has meant pupils not only remember the activities but also methods and are transferring knowledge to exam-based work. Time spent earlier on understanding is worthwhile. Pupils I still have contact with are transferring what they learnt the previous year into this year's work.

I see pupils gaining in confidence in maths and being willing to tackle more complex problems.

My pupils enjoy the lessons on the whole and are encouraged to think mathematically. They are engaged and motivated. They seem to have a deeper understanding of the concepts and think about applying these in different structures. They are prepared to try. The test results are better than those of parallel groups.

They are developing greater understanding than just through practising a few examples before moving to another topic. They ask their own questions; there is more discussion.

Students are tackling more extended tasks and give more thought to activities. Most students show a positive approach to their work. They seem to have more self-esteem. The discussions allow the students to express ideas orally rather than always having to write answers, which can be a barrier for some of them.

But are teachers' reports of 'feeling better' enough? We also looked at test results, progress in tackling an open-ended task, and observation data.

Test and task results

Each teacher had a different view about the value of tests, and the advantages and disadvantages of them, so we could not set up tests before and after the Project as there would have been no agreements about their style or even existence! Instead we used a range of kinds of test and task data as they became available. This means that we cannot make statements about results for all teachers and all students. However, here is a summary of the results we were able to collate without upsetting the normal practices of the teachers:

Comparative data

Five teachers provided data which compared their Year 7 test results to those from comparison classes. The classes were chosen from within their own schools, and were not always parallel classes in the sense that they had similar distributions of level scores at entry to secondary school. Because of the number of possible variables, and the small group sizes, we collated all the data from Project students rather than deal with individual schools[8].

1. In general test performance, Project students did roughly the same as comparison classes who had followed a more procedural, coverage-based, approach to the curriculum, using official guidelines. (In some schools they did better).

2. In questions which focused on mathematical thinking, such as requiring application of techniques and knowledge in an unusual way, or multi-stage reasoning, students in the Project groups *did significantly better than those in the comparison groups.*

Progress in tackling extended explorations and reasoning tasks

Students in the Project groups of the second year of the Project were given a mathematical situation to explore at the start of the year[9]. Information was gathered about the way this lesson went, the teachers' view of it, and the students' work. Work was graded using typical GCSE criteria for such work[10]. This task was repeated at the end of the year to see if students were more willing to engage in extended mathematical work, and showed more interest and initiative[11]. In all but a few individual students it was found that:

[8] One reason for this was to avoid having to take account of the wide range of factors describing the different kinds of comparison group: size, distribution of entry grades, experience of teacher and so on.

[9] The task used was taken from Ollerton (2002)

[10] The actual criteria used were those developed by the Association of Teachers of Mathematics for 100% coursework GCSE in 1988-1994. They are more detailed than those used by major examination boards, but fulfil the same standards.

[11] We used the same task because responses to open-ended, unstructured, tasks are always highly specific, and we wanted to compare before and after responses as closely as possible. The task provided potential for variation in the way it was explored, and there was little likelihood that students would recall previous answers during the post-test. However, if they did, then the marking criteria took into account repetitions.

Students showed considerable improvement in their willingness to work on the task, and produced more ideas, more lines of enquiry and were prepared to take more risks. Achievement was significantly higher at the end of the year.

This means that the students who might normally be expected to become more disaffected and reluctant to work were instead becoming more willing to work and more engaged. They showed significant improvement in the use of thinking skills in this task.

In addition, teachers reported more enthusiasm among students, and one class who had started the year noisily rejecting the task asked for it to extend over several lessons when it re-appeared at the end of the year.

Observation data about mathematical thinking

Although we had not managed to agree a definition of mathematical thinking, we all agreed that the following ways of thinking and working were important in mathematics, and that the students should be given opportunities to use them. One exciting result of the project was that students began to use these ways of thinking voluntarily elsewhere in their work, not just when teachers asked them to. We are not claiming that all students in all classes did all these things, but that among these low attaining students we saw several manifestations of sophisticated mathematical thinking, both prompted and unprompted:

Prompted	Unprompted
Choosing appropriate techniques	Choosing appropriate techniques
Contribute examples	Contribute examples
Describing connections with prior knowledge	Describing connections with prior knowledge
Finding similarities or differences beyond superficial appearance	Finding similarities or differences beyond superficial appearance
Generalising structure from diagrams or examples	Generalising structure from diagrams or examples
Identifying what can be changed	Identifying what can be changed
Making something more difficult	Making something more difficult
Making comparisons	Making extra kinds of comparison
Posing own questions	Generating own enquiry
Predicting problems	Predicting problems
Giving reasons	Giving reasons
Working on extended tasks over time	Asking to spend more time extending a task
Sharing own methods	Creating own methods and shortcuts
Using prior knowledge	Using prior knowledge
Dealing with unfamiliar problems	Initiating a mathematical idea
	Changing their mind with new experiences

These results repeat what has been found in several previous studies in which students are taught mathematics through a focus on thinking and understanding rather than focusing solely on coverage of content. However, previous studies have not focused specifically on the lowest achieving students, and they tend to focus on teachers who are using certain 'methods' or 'materials'. Thus this Project shows that teachers *who are free to innovate for themselves* are able to improve the attainment, engagement and mathematical thinking of low-achieving students. Their students were not disadvantaged in the short-term, their test results compared well to other groups, and there were significant gains in participation, self-esteem, and their ability and willingness to engage with extended, unfamiliar and complex tasks.

Students do at least as well as those in comparison groups on standard tests. Students do better than comparison groups on questions which involve mathematical thinking. Students' willingness and ability to take risks, and to engage with complex, open, extended tasks improves significantly.

Technical information about the research

The participants in this Project met at least twice a term to discuss what we felt was important, what our beliefs and intentions were, how we could construct tasks and teaching situations which enabled students to achieve in mathematics, and to share good ideas and practices which worked. Some of the teachers began to co-plan lessons and others found that the support of the group encouraged them to stick with changes they were making long enough to see results, whereas on their own they may have given up.

The research team did not tell teachers what to do, nor provide instructions about good tasks or good ways to organise teaching. Instead, teachers and researchers all put ideas into the discussions. What teachers did in classrooms was, therefore, their own ideas or adaptations and adoptions of ideas from others. The researchers' job was to provide support, ideas from experience and research literature, encouragement, and to collate data about practice. All teachers were provided with copies of recent books which contained appropriate classroom ideas from which to choose. Data were constantly reviewed and issues for discussion selected from it, so that similarities and differences between practices could be identified, aired, shared and discussed. The following kinds of data were collected and analysed:

- Records of meetings held one day a term for two years.
- Records of meetings in twilight time.
- Records of visits to schools and lesson observation by research officer, Els De Geest.
- Records of visits to schools by other academics, Anne Watson and Steph Prestage.
- Teachers' written notes about lesson plans and evaluations.
- Videos of lessons.
- Tape-recording and written notes of discussions sent to all Project members.

- Occasional written statements by teachers about beliefs and responses to particular issues.
- Some students' normal written work, including self-evaluation and tests where these were done.
- Written work relating to 'before' and 'after' tasks
- SATs, Progress and Optional tests where appropriate, including some from comparison groups.

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