Investigation of the Koch Snowflake

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Abstract: This paper explores the geometry and the behind one of the most famous fractals of all time – the Koch Snowflake. This paper also explores the concepts of infinity and limits, and how all these with the Koch Snowflake together lead to a beautiful result in the end.

Keywords: Geometry, fractals, Koch Snowflake, infinity, limits.

1 Introduction to the Koch Snowflake

The Koch Snowflake is a type of fractal; fractals are geometric figures that has a repeated pattern that goes on forever (usually approaches a limit of some value), and the patterns formed within it are self-similar. Meaning, when zooming into different parts of a fractal, the image is very similar to the image of the fractal itself. They are made by iterative processes. The Koch Snowflake is made as such:

1. Start by constructing an equilateral triangle with side length s. Let us call this figure K_0.
2. Divide each of the sides of the equilateral triangle into 3 equal parts, forming side lengths of $\frac{1}{3}s$.
3. Draw an equilateral triangle on each of the sides, with the base as the middle segment of lengths $\frac{1}{3}s$ from figure K_0 and erase the base segment. Let us call this newly formed figure K_1.
4. Repeat steps 2 and 3, but with different side lengths, namely, $\frac{1}{3^n}s$, where n is the number of times steps 2 and 3 are repeated. Thus, let K_n represent the Koch Snowflake after repeating steps 2 and 3 n times.

2 Perimeter of the Koch Snowflake

Let us now create a table showing the number of iterations (n), the side length (s_n), the number of sides (m_n), and the perimeter (p_n) of the Koch Snowflake using the information above:

Number of Iterations (n)	Side Length (s_n)	Number of Sides (m_n)	Perimeter (p_n)
0	1	3	3
1	1/3	12	4
2	1/9	48	16/3
3	1/27	192	64/9
4	1/81	768	256/27
5	1/243	3072	1024/81

Table 1: List of values for n, s, m, and p.

As we can see, the side length of K_n decreases by $\frac{2}{3}$ of the previous side length of K_n. The number of sides of K_n increases by 4 times the previous number of sides of K_n. The perimeter of K_n increases by $\frac{4}{3}$ times the pervious perimeter of K_n. Let us now express, algebraically, s_n, m_n, and p_n:

$$s_n = \frac{1}{3^n} \tag{2.1}$$

$$m_n = 3(4^n) \tag{2.2}$$

$$p_n = 3s(\frac{4}{3})^n \tag{2.3}$$

It is fairly obvious to see that the following limit converges to 0:

$$\lim_{n\to\infty} \frac{1}{3^n} = 0 \tag{2.4}$$

And that this limit diverges to infinity:

$$\lim_{n\to\infty} 3(4^n) = \infty \tag{2.5}$$

Note that this limit also diverges to infinity:

$$\lim_{n\to\infty} 3s_0(\frac{4}{3})^n = \infty \tag{2.6}$$

The perimeter grows exponentially as there are more iterations due to the fact that $\frac{4}{3}$ is greater 1, thus every time we multiply $\frac{4}{3}$ by itself, there is an increase in value.

3 Area of the Koch Snowflake

From the expressions above, we know that the number of sides multiplies by 4 after each iteration, and that the side length decreases by $\frac{2}{3}$ of the previous side length ($\frac{1}{3^n}$). So, the area of the smaller triangles formed after each iteration is $(\frac{1}{3})^2$, or $\frac{1}{9}$ of the area of the previous triangle.

Look at the table below:

Number of Iterations (n)	Number of Smaller Triangles Added (u_n)	Number of Sides (m_n)
0	0	3
1	3	12
2	12	48
3	48	192
4	192	768
5	768	3072

Table 2: List of values for n, u, and m.

Recall Eq.**(2.2)**, where: $m_n = 3(4^n)$, we can also apply this to the following observation: For every K_n, there are $3(4^{n\text{-}1})$ smaller triangles added to the previous Koch Snowflake, that is, for $n \geq 1$, which can be written like this:

$$u_n = 3(4^{n\text{-}1}) = \frac{3}{4} \times 4^n \tag{3.1}$$

Also recall that the area of the smaller triangles added is 1/9 of the previous area of triangles. So, to calculate the area that is added onto the original triangle after each iteration, we simply multiply u, $(\frac{1}{9})^n$, and A_0 (the original area of the triangle, which we define to be 1 here) together, as u determines the number of triangles that are added, and $(\frac{1}{9})^n$ determines the fraction of area of the new triangles that are formed after each iteration:

$$A_n = 3(4^{n\text{-}1}) \times (\frac{1}{9})^n \times A_0 \tag{3.2}$$

For example, if we were to find the area of, say, K_3, we can write this as:

$$A_3 = A_0 + [3(4^0) \times (\frac{1}{9})^1 \times A_0] + [3(4^1) \times (\frac{1}{9})^2 \times A_0] + [3(4^2) \times (\frac{1}{9})^3 \times A_0] + [3(4^3) \times (\frac{1}{9})^4 \times A_0] \tag{3.3}$$

Simply further and we get:

$$A_3 = A_0 + \frac{3(4^0)}{9^1}A_0 + \frac{3(4^1)}{9^2}A_0 + \frac{3(4^2)}{9^3}A_0 + \frac{3(4^3)}{9^4}A_0 \tag{3.4}$$

Here, we can factor out A_0 and $\frac{3}{9}$ separately, then simplify again to get:

$$A_3 = A_0(1 + \tfrac{1}{3}[(\tfrac{4}{9})^0 + (\tfrac{4}{9})^1 + (\tfrac{4}{9})^2 + (\tfrac{4}{9})^3]) \tag{3.5}$$

$$A_3 = A_0[1 + \tfrac{1}{3}\textstyle\sum_{k=0}^{3}(\tfrac{4}{9})^k] \tag{3.6}$$

Now, we can write the formula for finding the area of the Koch Snowflake – using Eq.**(3.6)** – after k iterations:

$$A_k = A_0[1 + \tfrac{1}{3}\textstyle\sum_{i=0}^{k}(\tfrac{4}{9})^i] \tag{3.7}$$

Let us now construct a table showing the number of iterations, number of smaller triangles added. area added, and the total area after k iterations of the Koch Snowflake:

Number of Iterations (n)	Number of Smaller Triangles Added (u_n)	Area Added (A_n)	Total Area (A_k)
0	0	0	1
1	3	1/3	4/3
2	12	4/27	40/27
3	48	16/243	376/243
4	192	64/2187	3448/2187
5	768	256/19683	31288/19683

Table 3: List of values for n, u_n, A_n, and A_k.

Let us now find the limit of the area of the Koch Snowflake, which is this expression here:

$$\lim_{k \to \infty} A_k = A_0[1 + \tfrac{1}{3}\textstyle\sum_{k=0}^{\infty}(\tfrac{4}{9})^k] \tag{3.8}$$

We shall now find the sum: $\sum_{i=0}^{k}(\frac{4}{9})^i$ and use it to calculate the limit of the area of the Koch Snowflake, given A_0. Geometric series are expressed as such, where a is some coefficient for all terms r, and r is some constant factor:

$$s = a + ar^1 + ar^2 + ar^3 + ar^4 + ar^5 + \ldots \tag{3.9}$$

Then, dividing by a on both sides, we get:

$$\frac{s}{a} = 1 + r^1 + r^2 + r^3 + r^4 + r^5 + \ldots \tag{3.10}$$

Multiplying by r on both sides, we get:

$$\frac{sr}{a} = r^1 + r^2 + r^3 + r^4 + r^5 + r^6 + \ldots \tag{3.11}$$

And by subtracting Eq.**(3.11)** from Eq.**(3.10)**, we get:

$$\frac{s - sr}{a} = 1 \tag{3.12}$$

All the other powers of r cancel out, leaving 1 behind. We then factor out s and multiply both sides by a to get:

$$s(1 - r) = a \tag{3.13}$$

Finally, we get the formula for s:

$$s = \frac{a}{1 - r} \tag{3.14}$$

Now, let us plug in values for a and r, then plug s into the infinite sum in Eq.**(3.8)**:

$$s = \frac{1}{1 - \frac{4}{9}} = \frac{1}{\frac{5}{9}} = \frac{9}{5} \tag{3.15}$$

$$\lim_{k \to \infty} A_k = A_0[1 + \frac{1}{3}(\frac{9}{5})] = A_0(1 + \frac{9}{15}) = A_0(1 + \frac{3}{5})) = A_0\frac{8}{5} \tag{3.16}$$

From here, we can see that the limit of the area of the Koch Snowflake as the number of iterations go to infinity, the result will be 8/5 of the original area of the triangle.