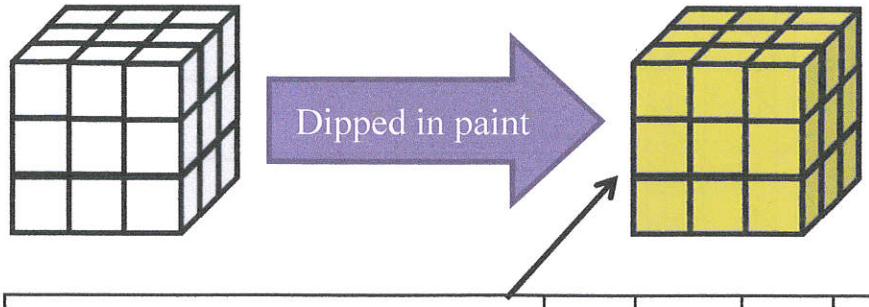
Mollie Bentley-Smith 8R

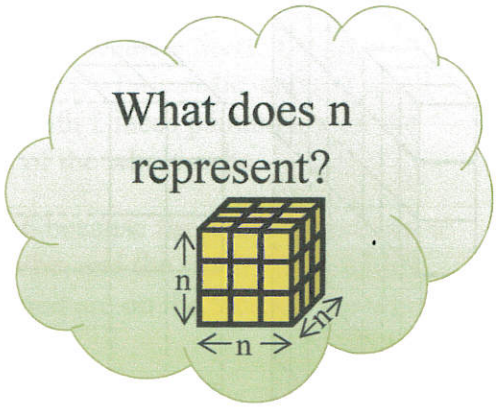
# Cube Problem

## The Problem:

A cube made up of small cubes is dipped into some paint. What are the $n^{th}$ terms for the number of small cubes with 6, 5, 4, 3, 2, 1 and 0 of their faces painted?

e.g.



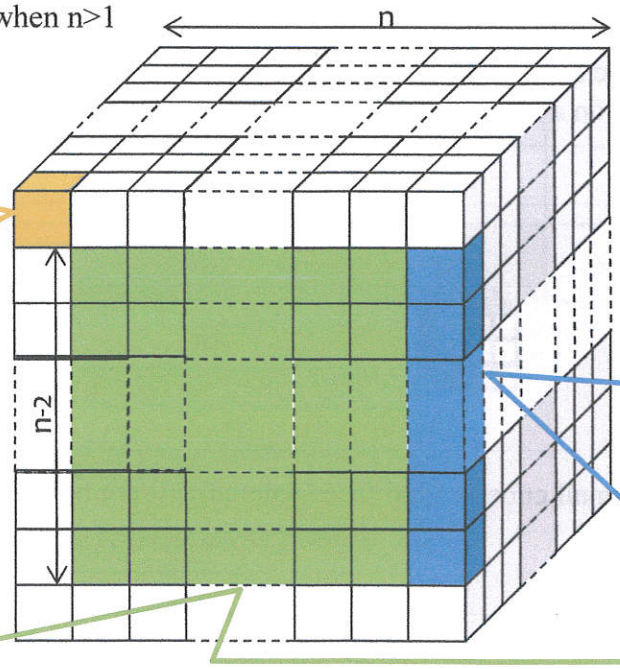

| Number of small cube faces painted | 6 | 5 | 4 | 3 | 2 | 1 | 0 |
|---|---|---|---|---|---|---|---|
| Number of small cubes | 0 | 0 | 0 | 8 | 12 | 6 | 1 |

## Assumptions and what we already know:

- When the cube is dipped into the paint, the paint only covers the surface area of the cube.
- We are counting the number of smaller cubes within the whole cube.
- The whole cube is completely made up of small cubes; there are no gaps or holes.
- **On a whole cube there are 8 corners**
- **On a whole cube there are 6 faces**
- **On a whole cube there are 12 edges**

## The formulas:

The Formulas only work when n>1




There is 1 small cube with three faces painted for each corner of the whole cube.

There are (n-2) small cubes with 2 faces painted for each edge of the whole cube. This is because 2 small cubes on each edge are corners, which have 3 of their faces painted. So, to get the number of small cubes on an edge that have 2 faces painted you must use the length of one edge (this dimension = n) and subtract 2.

There are $(n-2)^2$ small cubes with one face painted for each face of the whole cube. This is because on each face of the whole cube the small cubes round the edge either have 2 or 3 faces painted. This leaves a square of small cubes in the middle of each whole cube face which only have one face painted. To get the height of this square you must use the height of one face of the whole cube (this dimension = n) but subtract 2 from it to get rid of the edges and corners. All you then have to do is square it to get the number of small cubes in this middle square.

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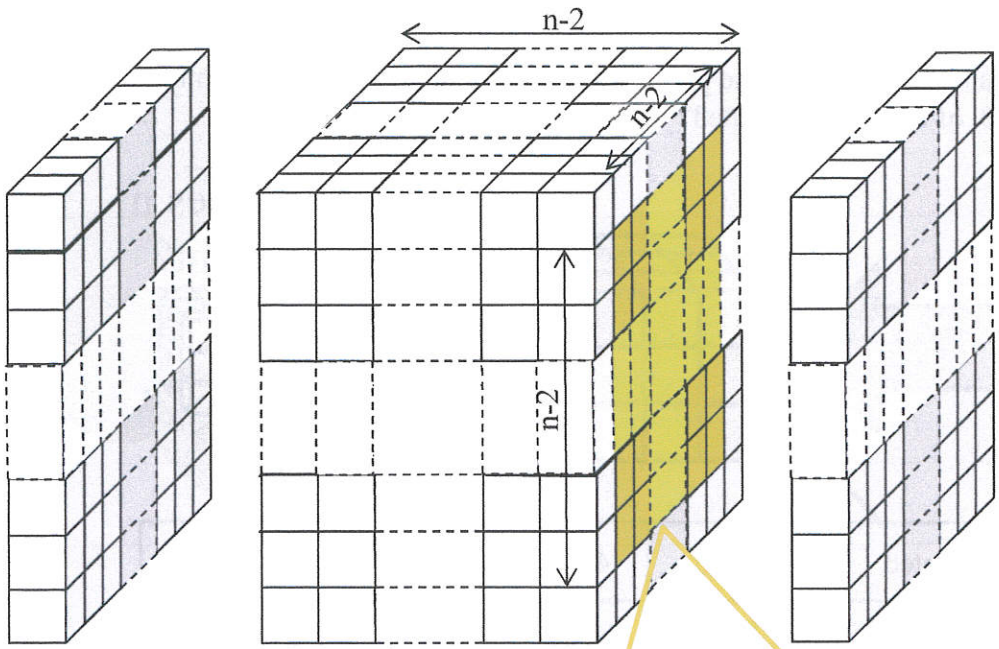
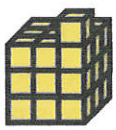
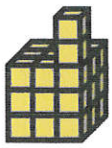
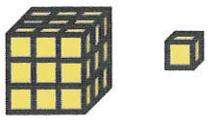



The only small cubes without any paint on them are the ones on the inside, beneath the outer layer of small cubes. These small cubes make up a cube. To get the height of this cube you must use the height of the whole cube (this dimension = n) but subtract 2 from it to get rid of the outer layer of small cubes. All you then have to do is cube it to get the number of small cubes in this middle cube.

The amount of small cubes with 6, 5 or 4 faces painted will always be 0. This is because, to make a cube, the small cubes must be joined to one another by at least 3 of their faces. This leaves a maximum of 3 faces that could be painted. If 4 or 5 faces on a small cube were painted, the shape of all the small cubes would not be a cube.

e.g.

 or  or 

The only case when the amount of small cubes with 6 faces painted will not be 0 is when n=1:

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| Number of small cube faces painted | $n^{th}$ term formula for the number of small cubes |
|---|---|
| 6 | 0 |
| 5 | 0 |
| 4 | 0 |
| 3 | 8 x 1 = 8 |
| 2 | 12(n-2) |
| 1 | $6(n-2)^2$ |
| 0 | $(n-2)^3$ |

1 x 8 because **there are 8 corners on a whole cube**, and the small cubes with 3 faces painted are on the corners of the whole cube.

(n-2) x 12 because **there are 12 edges on a whole cube**, and the small cubes with 2 faces painted are on the edges of the whole cube.

$(n-2)^2$ x 6 because **there are 6 faces on a whole cube**, and the small cubes with 1 face painted are on the faces of the whole cube.

## A proof that my formulas work:

All of the small cubes in the whole cube (this is $n^3$) will either have 6, 5, 4, 3, 2, 1 or 0 of their faces painted, so if you add up all the values for how many small cubes have 6, 5, 4, 3, 2, 1 and 0 of their faces painted, you should always get $n^3$.

| Number of small cubes | Number of small cube faces painted | | | | | | | Number of small cubes in the whole cube |
|---|---|---|---|---|---|---|---|---|
| | 6 | 5 | 4 | 3 | 2 | 1 | 0 | |
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | $= 1$ $1^3$ |
| 2 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | $= 8$ $2^3$ |
| 3 | 0 | 0 | 0 | 8 | 12 | 6 | 1 | $= 8+12+6+1 = 27$ $3^3$ |
| n | **0** | **0** | **0** | **8** | **12(n-2)** | **$6(n-2)^2$** | **$(n-2)^3$** | $8+12(n-2)+6(n-2)^2+(n-2)^3 =$ $n^3$ |

$8+12(n-2)+6(n-2)^2+(n-2)^3$
$= 8+(12n-24)+6(n-2)(n-2)+(n-2)(n-2)(n-2)$
$= 8+(12n-24)+6(n^2-4n+4)+(n^2-4n+4)(n-2)$
$= 8+(12n-24)+(6n^2-24n+24)+(n^3-6n^2+12n-8)$
$= 8+12n-24+6n^2-24n+24+n^3-6n^2+12n-8$
$= 12n-24+6n^2-24n+24+n^3-6n^2+12n$
$= 12n+6n^2-24n+n^3-6n^2+12n$
$= 12n24n-24n+n^3+12n$
$= 24n-24n+n^3$
$= n^3$