I started with putting in few odd values of $n$ into $9^n + 1^n$.

$$9^1 + 1^1 = 10,\ 9^3 + 1^3 = 73 * 10, 9^5 + 1^5 = 5905 * 10$$

Noticing that all of my result were a multiple of 10, I conjectured that $9^{2m-1} + 1^{2m-1} = 10p$, $m, p \in \mathbb{Z}^+$.

**Proof** (using induction)

As $1^{2m-1} = 1$ for all $m \in \mathbb{R}$, I only had to prove that $9^{2m-1} + 1$ was a multiple of 10.

Let $f(m) = 9^{2m-1} + 1$.

$f(1) = 9 + 1 = 10$, which is obviously a multiple of 10.

Now, assume that $f(m)$ is a multiple of 10.

$$f(m+1) = 9^{2m+3} + 1 = 81 * 9^{2m-1} + 1$$

$$f(m+1) - f(m) = 80 * 9^{2m-1}$$

$$f(m+1) = 80 * 9^{2m-1} + f(m)$$

Therefore, $f(m+1)$ is a multiple of 10.

If $f(m)$ is a multiple of 10 then $f(m+1)$ is shown to be a multiple of 10. As $f(1)$ is a multiple of 10, then $f(m)$ is a multiple of 10 for all $m \in \mathbb{Z}^+$.

Then, I noticed that each of the proposed problems were in the form $(10 - x)^n + x^n$ when $n$ was odd and in the form $(10 - x)^n - x^n$ when $n$ was even.

I considered if the general case, $(a - x)^n + x^n$, would be a multiple of $a$ when $n$ was odd and $(a - x)^n - x^n$ a multiple of $a$ when $n$ was even. After substituting in different values of $a$, $x$ and $n$ I decided my proposal was true and set about proving it.

**Proof**

Using the binomial expansion:

$$(a - x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(-x)^k$$

The last term is $(-x)^n$ .

If $n$ is odd then $(-x)^n = -(x^n)$ so adding $x^n$ to $(a - x)^n$ produces a multiple of $a$ as all of the other terms in the expansion share a factor of $a$.

But if $n$ is even then $(-x)^n = x^n$ so subtracting $x^n$ from $(a - x)^n$ produces a multiple of $a$ using the same logic as before.

Therefore $(a - x)^n + x^n$ is a multiple of $a$ when $n$ is odd, and $(a - x)^n - x^n$ is a multiple of $a$ when $n$ is even.