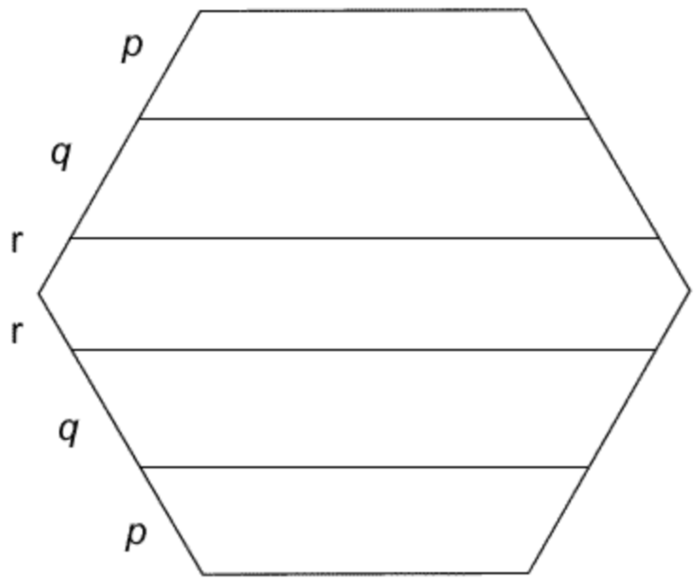
Hexagon Slices

Age 11 to 14 Short
Challenge Level ★★★

This regular hexagon has been divided into four trapezia and one hexagon. If each of the five sections has the same perimeter, what is the ratio of the lengths p, q and r?



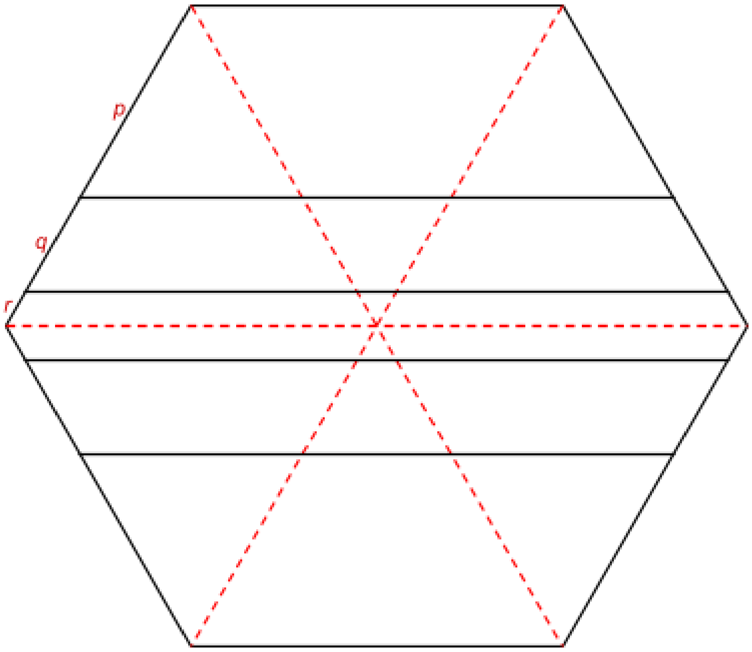
https://nrich.maths.org/5690

Original Solution: https://nrich.maths.org/5690/solution

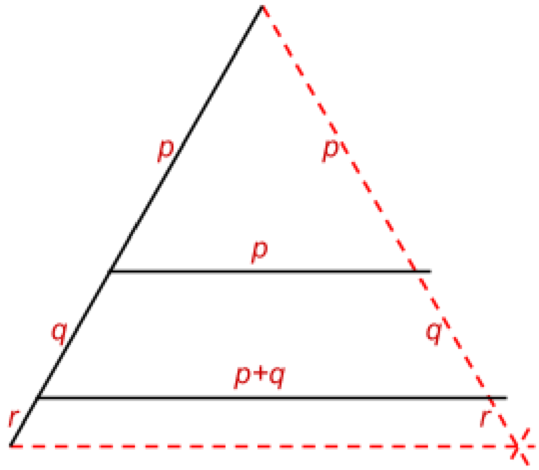
Additional Solutions by Canadian Academy, Class of 2024:

YoeEun Lee:

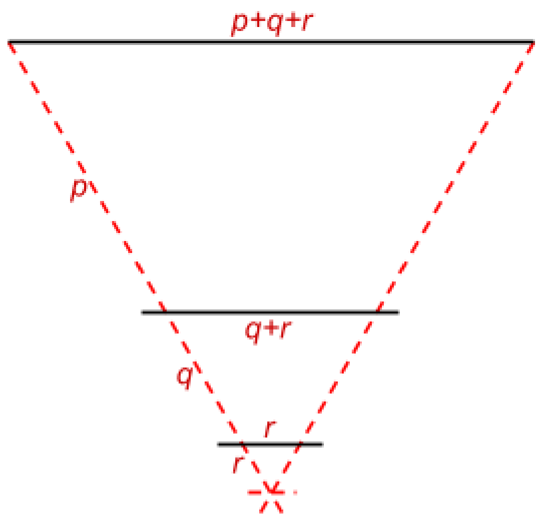
Divide the regular hexagon into six equilateral triangles.



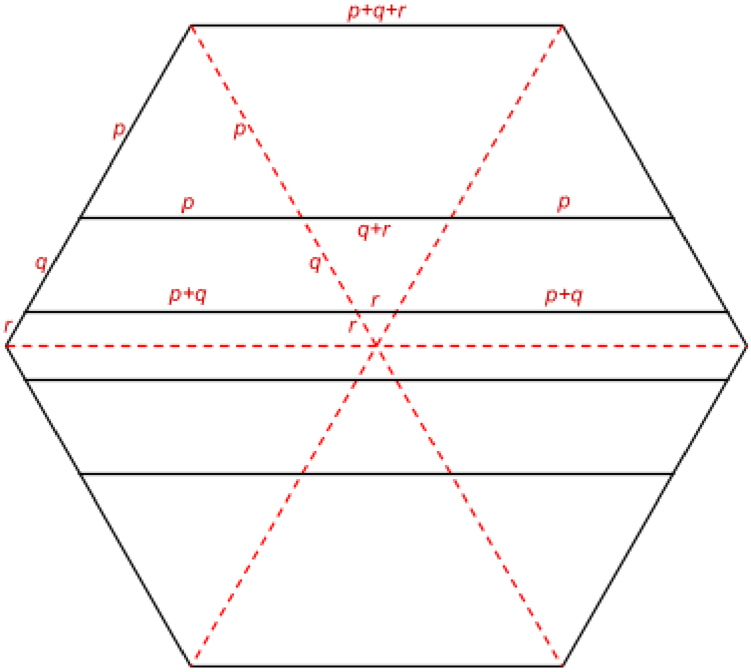
Considering the upper left equilateral triangle:



Then considering the upper middle equilateral triangle:



All triangles are equilateral and symmetric, so:



Then for the upper trapezoid,

$$P_1 = p + (p + q + r) + p + p + (q + r) + p = 5p + 2q + 2r$$

For the middle trapezoid,

$$P_2 = q + p + (q + r) + p + q + (p + q) + r + (p + q) = 4p + 5q + 2r$$

For the hexagon,

$$P_3 = [r + (p + q) + r + (p + q) + r] \times 2 = 4p + 4q + 6r$$

Since all the polygons have the same perimeter,

$$5p + 2q + 2r = 4p + 5q + 2r$$
$$p = 3q$$

and

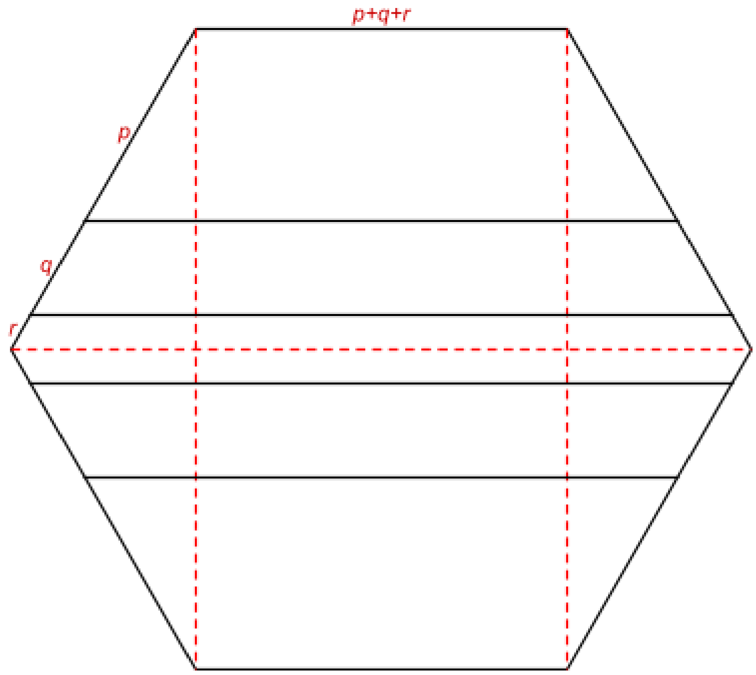
$$4p + 5q + 2r = 4p + 4q + 6r$$
$$q = 4r$$

Therefore,

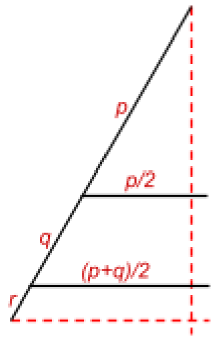
$$p : q : r = 3(4r) : 4r : r$$
$$p : q : r = 12 : 4 : 1$$

Jinwoo Son, Soyoon Park, Jinho Kim:

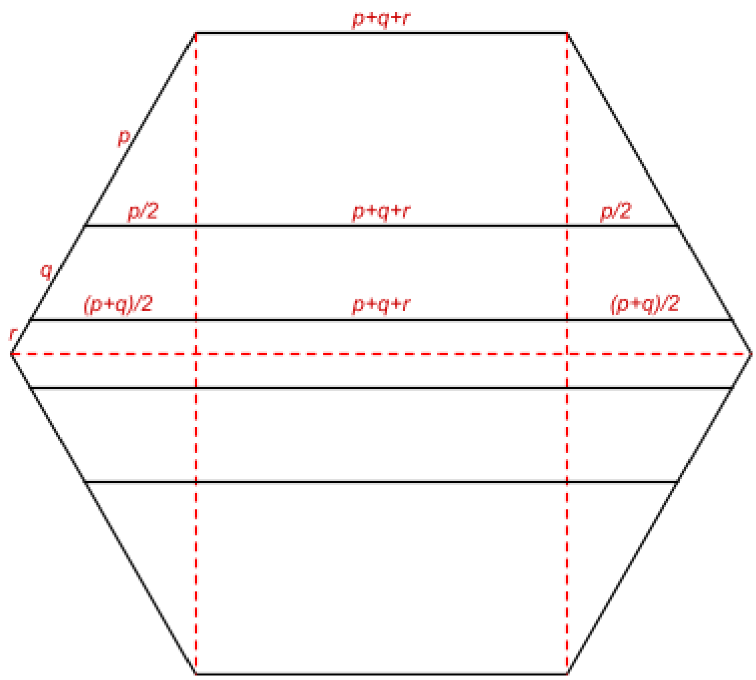
Divide the regular hexagon vertically into 30-60-90 right triangles:



Considering the upper left 30-60-90 right triangle (since the base is half its hypotenuse):



Then by symmetry:



Then for the upper trapezoid,

$$P_1 = p + (p + q + r) + p + \frac{1}{2}p + (p + q + r) + \frac{1}{2}p = 5p + 2q + 2r$$

For the middle trapezoid,

$$P_2 = q + \frac{1}{2}p + (p + q + r) + \frac{1}{2}p + q + \frac{1}{2}(p + q) + (p + q + r) + \frac{1}{2}(p + q)$$
$$= 4p + 5q + 2r$$

For the hexagon,

$$P_3 = \left[r + \frac{1}{2}(p + q) + (p + q + r) + \frac{1}{2}(p + q) + r\right] \times 2 = 4p + 4q + 6r$$

Since all the polygons have the same perimeter,

$$5p + 2q + 2r = 4p + 5q + 2r$$
$$p = 3q$$

and

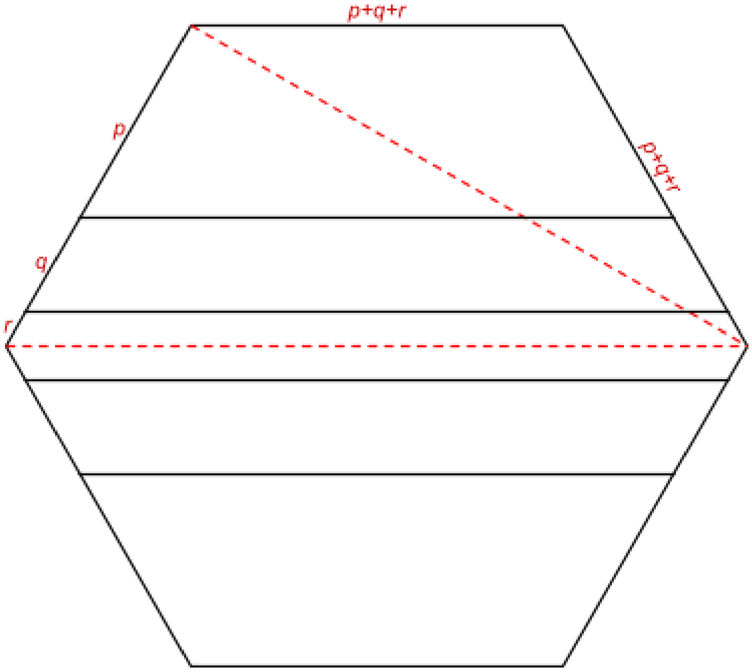
$$4p + 5q + 2r = 4p + 4q + 6r$$
$$q = 4r$$

Therefore,

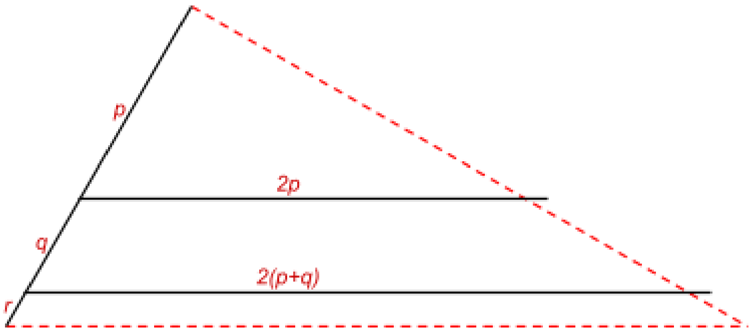
$$p : q : r = 3(4r) : 4r : r$$
$$p : q : r = 12 : 4 : 1$$

Kazuharu Nagamura, Devang Nair:

Dividing the regular hexagon into a large 30-60-90 right triangle and a large isosceles triangle (with the top and top-right sides having the same length):

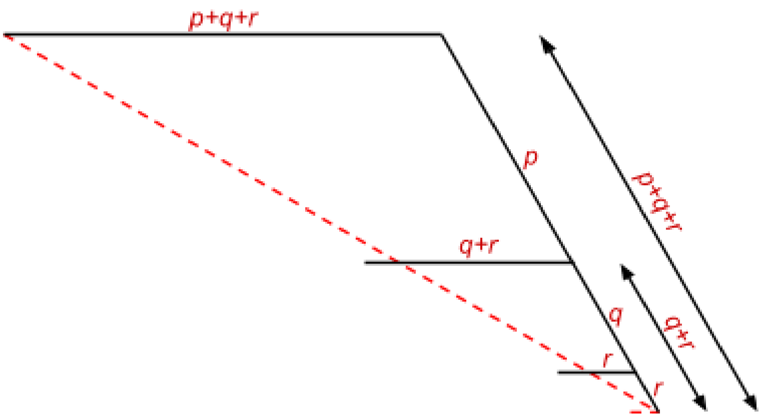


Considering the 30-60-90 right triangle and its interior similar triangles (because sides are parallel):



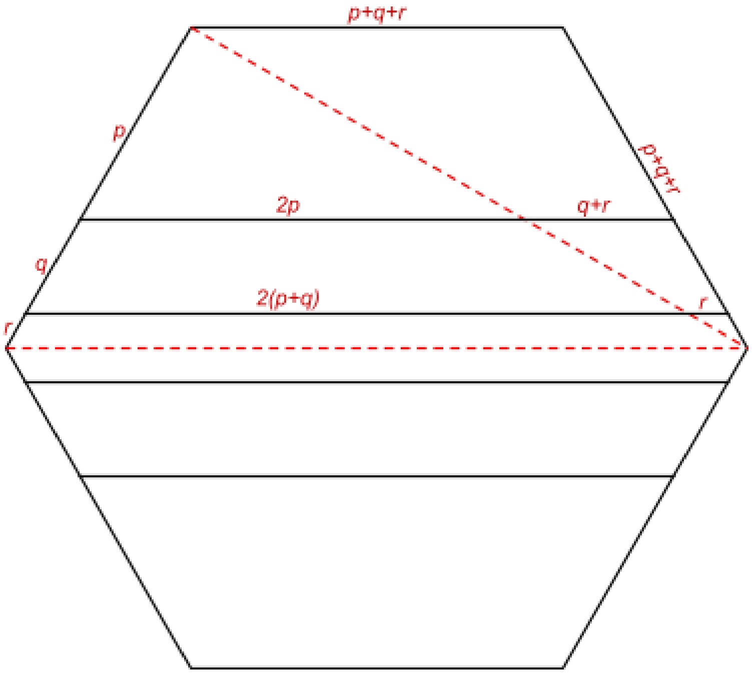
since the hypotenuse is twice its base.

Considering the large isosceles triangle and its interior similar triangles (because sides are parallel):



since the horizontal and right sides have equal lengths.

Then:



Then for the upper trapezoid,

$$P_1 = p + (p + q + r) + p + (q + r) + 2p = 5p + 2q + 2r$$

For the middle trapezoid,

$$P_2 = q + 2p + (q + r) + q + r + 2(p + q) = 4p + 5q + 2r$$

For the hexagon,

$$P_3 = [r + 2(p + q) + r + r] \times 2 = 4p + 4q + 6r$$

Since all the polygons have the same perimeter,

$$5p + 2q + 2r = 4p + 5q + 2r$$
$$p = 3q$$

and

$$4p + 5q + 2r = 4p + 4q + 6r$$
$$q = 4r$$

Therefore,

$$p : q : r = 3(4r) : 4r : r$$
$$p : q : r = 12 : 4 : 1$$