

③ Now, to both find out if it was possible to create any fraction from 0 and find more patterns, I wrote out how to get to a range of fractions which were between:  $0 < x \leq 1$ , since if they were greater (or less than), you could use T to add 1 to a fraction between  $0 < x \leq 1$

$$\frac{1}{2}, \frac{2}{2} = T^2 R T, T$$

$$\frac{1}{3}, \frac{2}{3}, \frac{3}{3} = T \left( \begin{matrix} T R T \\ T^2 R T^2 R T \end{matrix} \right)^2, T^3 R T, T$$

$$\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4} = T (T R T)^3, T^2 R T, T^4 R T, T$$

$$\frac{1}{5}, \frac{2}{5}, \dots, \frac{5}{5} = T (T R T)^4, T^3 R T^2 R T, T^2 R T^3 R T, T^5 R T, T$$

$$\frac{1}{6}, \frac{2}{6}, \dots, \frac{6}{6} = T (T R T)^5, T^2 R T^2 R T, T^2 R T, T^3 R T, T^6 R T, T$$

$$\frac{1}{7}, \frac{2}{7}, \dots, \frac{7}{7} = T (T R T)^6, T^3 R T^2 R T^2 R T, T^4 R T^2 R T, T^2 R T^2 R T^3 R T, T^2 R T^4 R T, T^7 R T, T$$

- This allows us to spot some patterns. We can already see  $\frac{1}{n}$  and  $\frac{n}{n!}$  (or  $\frac{n-1}{n!}$ ), which we have proved earlier.

- However, for  $\frac{n-2}{n}$ , there's a difference between when  $n$  is odd and

even. For every even ' $n$ ', the combinations slightly change:

$$\left( \frac{2}{4} \rightarrow \frac{4}{4} \right) = T^2 R T \rightarrow T^3 R T, \text{ the first } T \text{ increases by } 1.$$

$\frac{6}{6} \rightarrow$  To find this 'power' of T, you do  $\frac{n}{2}$

- This gives you  $T^{\frac{n}{2}} R T = \frac{n-2}{n}$

Proof this works for all even numbers.

$$0 \xrightarrow{T^{\frac{n}{2}}} \frac{n}{2} \xrightarrow{R} \frac{-2}{n} \xrightarrow{T} \frac{n-2}{n}$$

However this only works for even numbers, as otherwise,  $n$  would not give a whole number, which is required (T can only be performed a whole number of times).

For every odd 'n', the combination also slightly changes:

$$\left(\frac{1}{3} \rightarrow \frac{3}{5} \rightarrow \frac{5}{7}\right) = T^2 R T^2 R T \rightarrow T^2 R T^3 R T \rightarrow T^2 R T^4 R T$$

- Here's how the power on the second T set relate:

power { 2, 3, 4 } → To get from the number (n), to the power, put  
 number { 3, 5, 7 } it in  $\frac{n+1}{2}$

Proof this works for all odd numbers

$$0 \xrightarrow{T^2} \frac{1}{2} \xrightarrow{T^{\frac{n+1}{2}}} \frac{-1}{2} \xrightarrow{R} \frac{-1 + \frac{n+1}{2}}{2} = \frac{n-2}{2} \xrightarrow{T} \frac{n-2}{n}$$

However, this method only works for  $\frac{n-2}{n}$ . Ideally, we would

$$\text{be able to perform } 0 \xrightarrow{T^m} \frac{n}{n} \xrightarrow{R} \frac{-m}{n} \xrightarrow{T} \frac{n-m}{n}$$

but  $\frac{n-m}{n}$  isn't always a whole number.



If you look again at the fractions, you can notice something else

$$\frac{n-2}{n} = T^2 R T^3 R T = (T^2 R T^2) (T R T)$$

↑  $\frac{3}{2}$

$$\frac{n-3}{n} = T^2 R T^2 R T^2 R T = (T^2 R T^2 R T) (T R T)$$

↑  $\frac{1}{3}$

$$\frac{n-4}{n} = T^4 R T^2 R T = (T^4 R T) (T R T)$$

↑  $\frac{3}{4}$

- Each set of functions can be split into 2 parts: Part 2 = 'TRT' and Part 1 =  $\frac{n-m}{m}$ , where  $m$  is in ' $\frac{n-4}{4}$ '

- You can split Part one further, into a fractional component and a whole number component e.g.

$$\frac{n-3}{n} = T^2 R T^2 R T^3 R T = (T^2 R T^2 R T) (T) (T R T)$$

↑  $\frac{1}{3}$                       ↑  $+1$

Whole number components can easily be added by adding 'T' → To prove that you can make a set of  $n-m$  fractions, you must prove that you can make  $\lfloor n \rfloor$   
 $\frac{1}{m}, \frac{2}{m}, \frac{3}{m}, \dots, \frac{m}{m}$

Let's have a look at  $\frac{1}{2}$ :

$$\frac{n-1}{n} = T^2 R T. \text{ Because you can make } \frac{1}{2}, \text{ you have proved that}$$

you can make any fraction  $\frac{n-2}{n}$ , because the fractional remainder of  $\frac{n-2}{2}$  is always  $\frac{1}{2}$  (or 0/1)

- Using this, you can show that you can make  $\frac{1}{3}$ :  
 $\left(\frac{n-2}{3} = \frac{1}{3}\right)$ . This allows you to create any fraction  $\frac{n-3}{n}$ , as fractional remainders of  $\frac{n-3}{3}$  are  $\frac{1}{3}, \frac{2}{3}$  (or 0/1).

- This proves  $\frac{1}{4}$  as  $\frac{n-3}{n}$  is possible, allowing the cycle to continue. Proved since all fractions  $\frac{n-1}{n}$  are proved, as shown earlier.

Overall

- To prove you can make any fraction, you must prove that you can make any  $\frac{n}{m}$  as  $\frac{n}{m} = \frac{n-m}{n}$

-  $\frac{n}{m}$  can be split up into an integer and a fraction  
 $\rightarrow$  Integers can be added by using 'T'  
 $\rightarrow$  You must prove you can make the fraction, which will be from  $\frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}$

- You can make  $\frac{n}{1}$  using 'T'<sup>n</sup>  $\rightarrow$  this proves you can have  $m=1$  and make  $\frac{n-1}{n}$ , or  $\frac{2-1}{2} = \frac{1}{2}$

- Because you know how to make  $\frac{1}{2}$ , you can make any  $\frac{n}{2}$  because the only fractional remainder is  $\frac{1}{2}$   
 $\rightarrow m=2; \frac{n-2}{n} \rightarrow$  can make  $\frac{3-2}{3} = \frac{1}{3}$  and  $\frac{2-1}{3} = \frac{1}{3}$



- Using this, you can make all possible fractional components of  $\frac{n}{3}$  ( $\frac{1}{3}$  and  $\frac{2}{3}$ ) proving  $\frac{n-3}{n}$
- This proof proves even more fractional components, allowing  $\frac{n-4}{n}$
- Continue, proving  $\frac{n-m}{n}$
- Proof of  $\frac{n-m}{n}$  means you can make any fraction from  $\frac{1}{n}$  to  $\frac{n}{n}$
- Adding  $T$  therefore allows you to create any positive fraction
- This means that by using  $R$ , you can find the negative reciprocal of a fraction. If you know all positive fractions, you know all negative fractions.