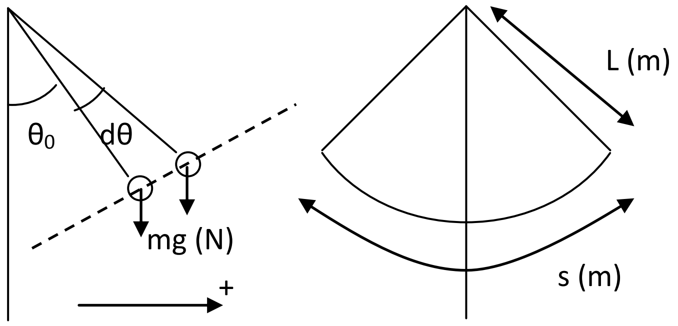
Not so simple pendulum 1

Pre-requisites:

- Physics or M1 Uniform equations of motion and Newton's 1^{st}, 2^{nd} and 3^{rd} laws
- C3 calculus
- C1 circle bits

To start solving this problem imagine a bung already at an angle and then compare this to what it looks like a fraction of a second later – from this we can resolve forces.



The first image is of the bung in 2 places a fraction of a second apart, and as you can see I have added a line that the bung passes through in both frames. This is to show that the bung moves in parallel to the line and therefore this line is a tangent to the path of the bung. We can therefore resolve the forces in the direction of the line and perpendicular to it. The force along the tangent will be $F=-mg\sin(\theta)$, where F is the force acting on the bung, m is the mass of the bung, θ is the angle and g is the gravitational field strength. It's negative because the force is acting in the opposite direction to its position. The force acting perpendicular to the tangent will be equal to the tension in the string so we don't need to know about it. As everyone knows, F=ma and the acceleration is the rate of change of the velocity, and therefore the rate of change of the rate of change of the displacement. Then applying these equations to the tangential force gives:

$$\frac{d^2s}{dt^2} = -g\sin(\theta)$$

Where s is the displacement. Now if you look at the picture on the right you can see that $s=\theta L$ so now the equation becomes:

$$L\frac{d^2\theta}{dt^2} = -g\sin(\theta)$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta)$$

Which is in the form that was wanted and means that $k^2 = \frac{g}{L}$ and therefore k is measured in Hertz. This also shows that the angle is independent of how much the bung weighs just as you would expect. Re-arranging and substituting you get:

$$L \approx \frac{9.81}{k^2}$$

Which can be used to work out how long the string would have to be for certain values of k – ie k=0.1Hz, L=981m. k=1Hz, L=9.81m. k=10Hz, L=0.0981m. From this I think that k might represent the frequency of oscillation of the bung because the longer the length of string the slower the frequency, just as you would expect. For solving the differential equation, consider the Taylor expansion of the sin(x) is:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Which shows that for small θ, sin(θ) is approximately θ as the other terms are too small to affect the result. Using this assumption the differential equation becomes:

$$\frac{d^2\theta}{dt^2} \approx -\frac{g}{L}\theta$$

Which can easily be solved using the assumption:

$$\theta = A\sin(\omega t) + B\cos(\omega t)$$

To give the solution for the governing function of the bung:

$$A = 0, B = \theta_0, \omega = k = \sqrt{\frac{g}{L}}$$

$$\theta = \theta_0 \cos\left(t\sqrt{\frac{g}{L}}\right)$$

Now the time period of this equation is equal to the time taken for the bung to go round once and get back to the same point, so you need 2 values of t for which θ has the same value. As you can see this will happen when the inside of the cosine function is equal to 0 and then 2 pi.

$$0 = t\sqrt{\frac{g}{L}}$$

$$\therefore t = 0$$

$$2\pi = t\sqrt{\frac{g}{L}}$$

$$\therefore t = 2\pi\sqrt{\frac{L}{g}}$$

Therefore the time period is:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

You can get the same result by thinking of k as the angular velocity instead:

$$k = \omega = \frac{2\pi}{T} = 2\pi f$$

$$\therefore k \propto f$$

This also shows that my idea of k being the frequency of oscillation was kind of correct as they are directly proportional. This also shows that the time period is:

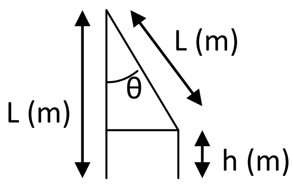
$$T = \frac{2\pi}{k} = 2\pi\sqrt{\frac{L}{g}}$$

And therefore the time of a quarter of the oscillation is:

$$\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{L}{g}}$$

Now if we consider a bung on the end of a string starting at an angle small enough to satisfy the assumption about θ, the kinetic energy of the bung after a quarter of the oscillation should be exactly the same as a separate bung in free-fall for the same vertical distance. To work out the kinetic energy of the system I must first find the linear velocity by multiplying the angular velocity by the radius, or in other words multiply the rate of change of the angle by the length of the string:

$$v = \omega r$$
$$v = \frac{d\theta}{dt} L$$
The linear velocity is therefore:
$$|v| = \theta_0 \sin\left(t\sqrt{\frac{g}{L}}\right)\sqrt{Lg}$$
Substituting in for t the value of a quarter oscillation and you get:
$$|v| = \theta_0 \sin\left(\frac{\pi}{2}\sqrt{\frac{L}{g}}\sqrt{\frac{g}{L}}\right)\sqrt{Lg}$$
$$|v| = \theta_0 \sin\left(\frac{\pi}{2}\right)\sqrt{Lg}$$
$$|v| = \theta_0\sqrt{Lg}$$
Which can then be used to determine the kinetic energy:
$$\text{KE} = \frac{1}{2} m v^2$$
$$\text{KE} = \frac{1}{2} m L g \theta_0^2$$
Now we need to work out the kinetic energy of the bung in free fall.



In this picture h is the distance the bung will fall and by using trigonometry you can see that h=L-Lcos(θ)=L(1-cos(θ)). Now by using the uniform equations of motion we can work out the kinetic energy of the bung:
$$v^2 = u^2 + 2as$$
$$v^2 = 2gh$$
$$v^2 = 2Lg(1 - \cos(\theta_0))$$
$$\text{KE} = \frac{1}{2} m v^2$$
$$\text{KE} = mLg(1 - \cos(\theta_0))$$
This could also have been done by considering the change in gravitational potential energy. Now we have equations for the kinetic energy of both bungs after falling the same distance, and if we set them equal to each other we can see that some of the terms can cancel and you end up with:
$$1 - \cos(\theta_0) = \frac{\theta_0^2}{2}$$
$$\cos(\theta_0) = 1 - \frac{\theta_0^2}{2}$$
Which are the first 2 terms of the Taylor expansion of the cosine function. Now because we assumed the time period to be correct while working this out and got a valid result this proves that as long as the starting angle is relatively small and obeys the sine or cosine Taylor expansion then the predicted time period will be correct.