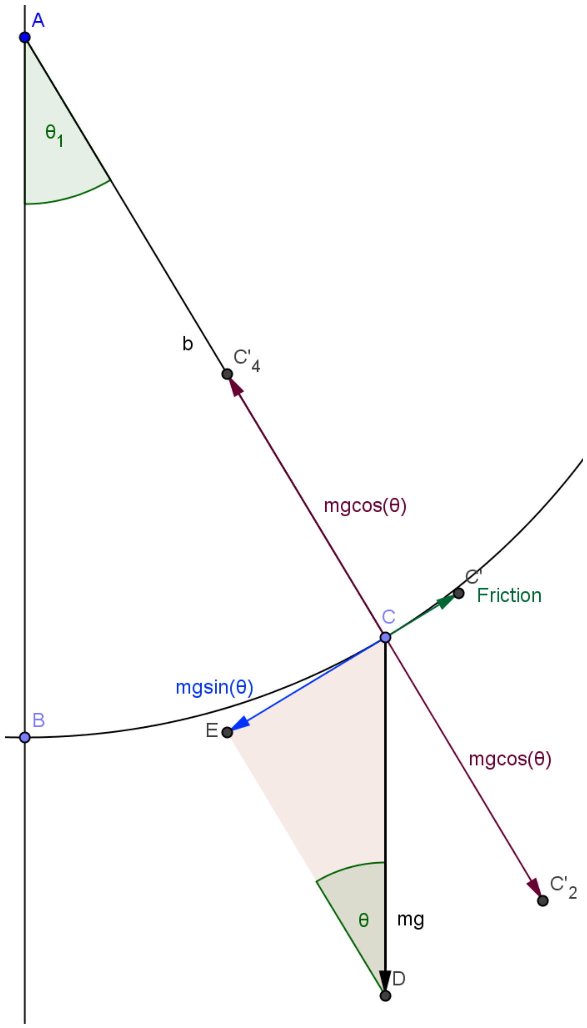


FIGURE 1 (not to scale)

Considering the simple pendulum with friction (Letting y = θ):

This consists of a point mass attached to an inextensible string. A force acting on the point mass is its weight. This appears as a black vector in the diagram. Assuming upwards as positive:

$$W = -mg$$

Where m = mass, g = gravitational acceleration.

The weight can be split into two components: the tangential force and the force acting away from the centre of rotation. The latter force, shown in brown, is balanced by the reaction force acting towards the centre and so can be ignored. Using trigonometry for right angled triangle *CDE*, the tangential force acting on the bob is

$$F_g = -mgsin(y)$$

The force due to friction acts tangentially on the bob. Since it acts in the opposite direction to the motion, the coefficient has to be negative. Keeping the friction coefficient constant allows the differential equation to be linear and so, easy to solve. Making the frictional force proportional to the tangential velocity *v* rather than to the angular velocity ω prevents the pendulum's length from affecting the frictional force.

$$F_f = -\lambda \frac{ds}{dt}$$

where s = displacement as a vector tangential to the pendulum's arc of motion

If y is measured in radians, L = arc length, r = pendulum length:

$$\frac{dy}{dt} = \frac{1}{r}\frac{dL}{dt} = \frac{1}{r}\frac{ds}{dt}$$

It follows that:

$$\frac{d^2y}{dt^2} = \frac{1}{r}\frac{d^2L}{dt^2} = \frac{1}{r}\frac{d^2s}{dt^2}$$

Letting the resultant force acting on the bob be F, Newton's second law of motion states that

Force = mass * linear acceleration

$$F = m\frac{d^2s}{dt^2}$$

$$m\frac{d^2s}{dt^2} = -mgsin(y) - \lambda\frac{ds}{dt}$$

$$\frac{m\frac{d^2s}{dt^2}}{r} = -\frac{mgsin(y)}{r} - \frac{\lambda\frac{ds}{dt}}{r}$$

Equivalent to

$$m\frac{d^2y}{dt^2} = -\frac{mgsin(y)}{r} - \lambda\frac{dy}{dt}$$

$$m\frac{d^2y}{dt^2} + \lambda\frac{dy}{dt} + \frac{mgsin(y)}{r} = 0$$

Now, since y is measured in radians, for small values of y, the following approximation holds:

$$y \approx \sin(y)$$

When y = 20° = π/9 rad the value of 100%*(y – sin(y))/sin(y) = 2.06% error, so the approximation is suitable to use for angles less than 20°. This serves to make the differential equation linear and so easy to solve.

Therefore

$$m\frac{d^2y}{dt^2} + \lambda\frac{dy}{dt} + \frac{mgy}{r} = \mathbf{0}$$

$$m\frac{d^2y}{dt^2} + \lambda\frac{dy}{dt} + k^2y = \mathbf{0}$$

Where $k = \sqrt{\frac{mg}{r}}$

Unit of k is $\sqrt{N\ m^{-1}}$

Now here is the given solution. I will differentiate it to obtain the original differential equation.

$$y = e^{jt}\left(Acos(\boldsymbol{h}t) + \ Bs\mathbf{in}(ht)\right)$$

$$j = \frac{-\lambda}{2m} \qquad \boldsymbol{h} = \sqrt{\frac{k^2}{m} - \frac{\epsilon^2}{4m^2}}$$

Note: I included the B constant to elaborate a point later in my explanation. In the given solution and in the rest of this particular proof, B = 1 and could otherwise be omitted.

Differentiating y using the product rule and chain rule:

$$\frac{dy}{dt} = e^{jt}\left(j\left(Acos(\boldsymbol{h}t) + \ Bsin(\boldsymbol{h}t)\right) + \ \boldsymbol{h}\left(Bcos(\boldsymbol{h}t) - \ Asin(ht)\right)\right)$$

Let $\alpha \ = Acos(ht) + \ Bsin(ht)$

Let $\beta = \ Bcos(\boldsymbol{h}t) - \ Asin(ht)$

Finding the derivative of α:

$$\frac{d\alpha}{dt} = \boldsymbol{h}\left(Bcos(\boldsymbol{h}t) - \ Asin(\boldsymbol{h}t)\right) = \boldsymbol{h}\beta$$

Finding the second derivative of α:

$$\frac{d^2\alpha}{dt^2} = \ -\boldsymbol{h}^2\left(Acos(\boldsymbol{h}t) + \ Bsin(\boldsymbol{h}t)\right) = \ -\boldsymbol{h}^2\alpha$$

Here I noticed that α was behaving like the exponential function e^{ct} where c is an arbitrary constant. Substituting $\alpha = \ e^{ct}$ we need to find c:

$$\frac{d^2}{dt^2}\ e^{ct} = c^2e^{ct} = c^2\alpha$$

$$c^2 = \ -\boldsymbol{h}^2$$

$c = \boldsymbol{h}i$

$\alpha = e^{\boldsymbol{h}it}$

Consequently, using the first derivative of α:

$\boldsymbol{h}i\, e^{\boldsymbol{h}it} = \boldsymbol{h}\beta$

$\beta = ie^{\boldsymbol{h}it}$

Substituting the exponential forms of α and β into the first derivative of y:

$$\frac{dy}{dt} = e^{jt}\left(je^{\boldsymbol{h}it} + \ \boldsymbol{h}ie^{\boldsymbol{h}it}\ \right)$$

$$\frac{dy}{dt} = e^{jt}e^{\boldsymbol{h}it}(j + \boldsymbol{h}i)$$

$$\frac{dy}{dt} = (j + \boldsymbol{h}i)e^{(j+\boldsymbol{h}i)t}$$

Hence, obtaining the second derivative of y is very easy

$$\frac{d^{\mathbf{2}}y}{dt^{\mathbf{2}}} = (j + \boldsymbol{h}i)^{\mathbf{2}}e^{(j+\boldsymbol{h}i)t}$$

Rewriting y in exponential form by substituting $\alpha = e^{\boldsymbol{h}it}$

$$y = e^{(j+\boldsymbol{h}i)t}$$

So we can form a linear differential equation with constant coefficients a, b and c:

$$a\frac{d^{\mathbf{2}}y}{dt^{\mathbf{2}}} + b\frac{dy}{dt} + cy = \mathbf{0}$$

Substituting

$$a(j + \boldsymbol{h}i)^{\mathbf{2}}e^{(j+\boldsymbol{h}i)t} + b(j + \boldsymbol{h}i)e^{(j+\boldsymbol{h}i)t} + ce^{(j+\boldsymbol{h}i)t} = \mathbf{0}$$

$$a(j + \boldsymbol{h}i)^{\mathbf{2}} + b(j + \boldsymbol{h}i) + c = \mathbf{0}$$

This is a quadratic equation with a positive root of j + hi. Similarly, a negative root, j – hi, exists, but since here we have the positive root, I will use the positive form of the quadratic formula.

$$j + \boldsymbol{h}i = \frac{-b + \sqrt{b^{\mathbf{2}} - \mathbf{4}ac}}{\mathbf{2}a}$$

Substituting the values of j and h

$$\frac{-\lambda}{2m} + \sqrt{\frac{k^2}{m} - \frac{\epsilon^2}{4m^2}}\, i = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-\lambda}{2m} + \sqrt{\frac{\epsilon^2}{4m^2} - \frac{k^2}{m}} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{-\lambda + \sqrt{\epsilon^2 - 4mk^2}}{2m} = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

From here it emerges that $\epsilon = \lambda$ and that

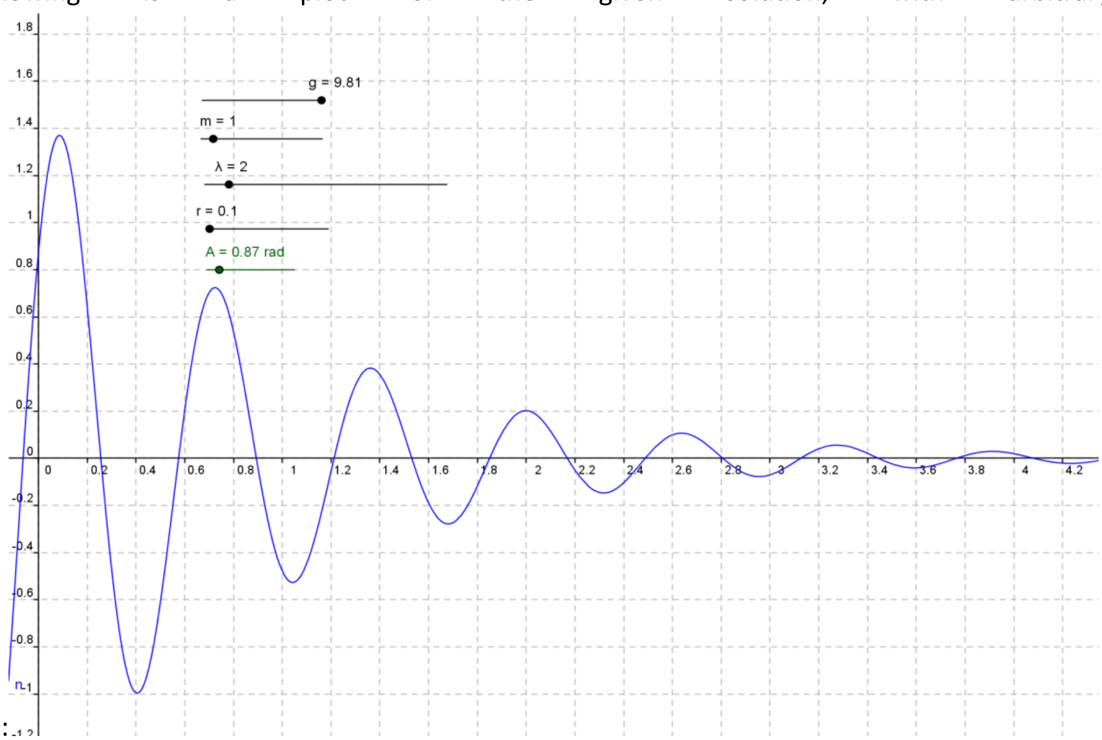
$a = m$, $b = \lambda$, $c = k^2$

Q.E.D.

Investigating the solution:

It appears that regardless of the values of A and B, the solution satisfies the differential equation. So A and B are constants of integration.

The following is a plot of the given solution, with arbitrary parameters:



Here one notices that the equation is a good model of friction as the amplitude of the oscillations decays over time. The frequency also drops since energy is being lost from the system.

However I would like the function to take into consideration the initial amplitude of the oscillation and also the initial velocity. This is where the constants A and B come in.

Assume that the pendulum is released **from rest** and at **maximum amplitude**.

$y(0) = a = \text{ initial amplitude}$

$y(0) = e^{j*0}(A\cos(h*0) + B\sin(h*0))$

$a = A$

Now to incorporate the initial velocity:

$y'(0) = 0$

$y'(t) = e^{gt}\left(j(a\cos(ht) + B\sin(ht)) + h(B\cos(ht) - a\sin(ht))\right)$

$y'(0) = aj + hB$

$0 = aj + hB$

$B = -\frac{aj}{h}$

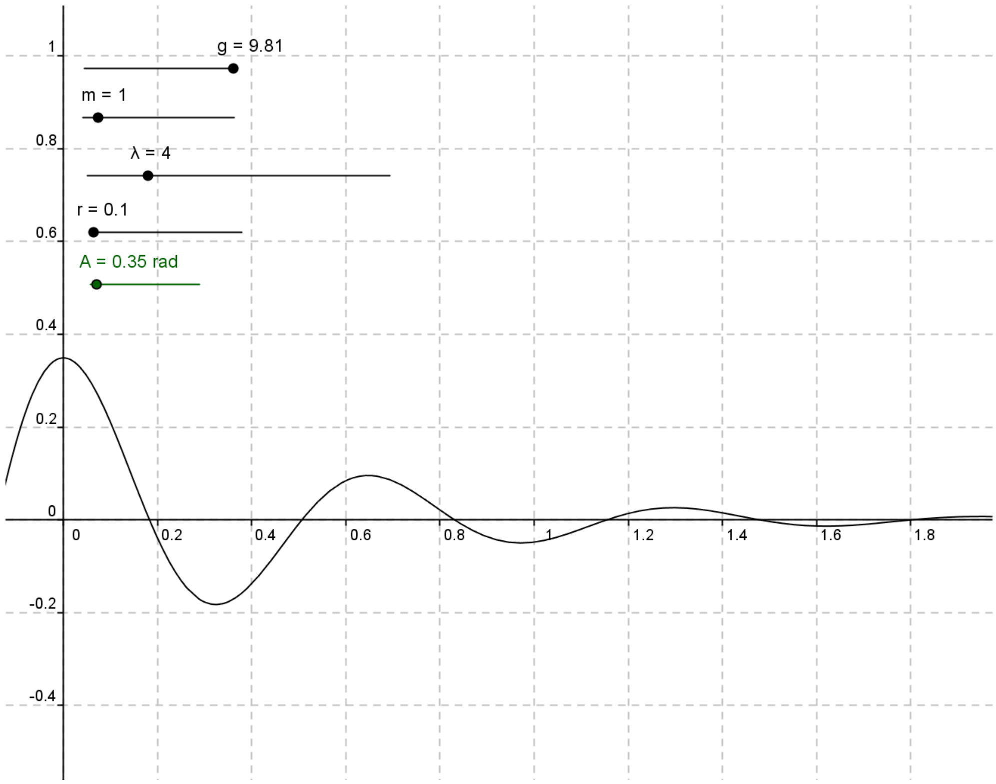
Substituting A and B into initial equation gives

$y = ae^{\frac{-\lambda}{2m}t}\left(\cos(ht) + \frac{\lambda}{2mh}\sin(ht)\right)$

$$h = \sqrt{\frac{k^2}{m} - \frac{\epsilon^2}{4m^2}}$$

Where *a* is initial amplitude.

Plotting this gives a more manageable graph, e.g this plot is for a = 0.35 rad, $\lambda = 4$



The range of λ for which the above equation applies

As shown in the proof, this particular equation applies as long as the solutions of $\frac{-\lambda + \sqrt{\lambda^2 - 4mk^2}}{2m}$ are complex. This means that $\lambda^2 - 4mk^2 < 0$. Since $\lambda \geq 0$ then $0 \leq \lambda < 2k\sqrt{m}$