Vector Walk

By Michael Slack

$$b_1 = \binom{2}{1}, b_2 = \binom{0}{1},$$

First I note that $(b_1 - b_2) = \binom{2}{0}$. This means that by using any combination of $(b_1 - b_2)$ and b_2, I can reach any pair of coordinates of the form $(2m, n)$, where m and n are integers.

To find another pair of vectors that can reach these same coordinates, I note that the two vectors must reduce to the same pair of x- and y-vectors. For example,

$$c_1 = \binom{4}{1}, c_2 = \binom{2}{1},$$

works since

$$c_x = (c_1 - c_2) = \binom{2}{0}, c_y = 2c_2 - c_1 = \binom{0}{1},$$

and there is no possible smaller c_x or c_y to be found. So by using any combination of c_x and c_y, I can reach any coordinates of the form $(2m, n)$.

Finding vector walks that never coincide with the vector walk b

There are a few ideas we can consider to achieve this.

One idea would be to start at the coordinates $(1,0)$ and use the vectors b_1 and b_2. This walk would reach any pair of coordinates of the form $(2m + 1, n)$ using my earlier workings. Since this walk would always have odd x-coordinates whilst the original walk always had even x-coordinates, the two walks could never coincide.

Another idea would be to find a pair of vectors that reduce to a different pair of x- and y-vectors. If we chose integer spacing in either or both of the x and y directions, however, this wouldn't work. If your reduced x-vector was $\binom{i}{0}$ and your reduced y-vector was $\binom{0}{j}$, then your new vector walk would certainly coincide with b at $(2i, j)$. Using rational spacing wouldn't work either; if your reduced vectors were $\binom{i/j}{0}$ and $\binom{0}{p/q}$, then you would get coincidence at $(2i, p)$ after having taken 2j x-vectors and q y-vectors.

This leaves open the possibility of using irrational spacing. For example, if your new vector walk reduced to $\binom{\sqrt{2}}{0}$ and $\binom{0}{\sqrt{3}}$, then you could be certain of no coincidence other than at the origin since there is no other crossover between the general coordinates $(2m, n)$ and $(\sqrt{2}p, \sqrt{3}q)$ (bearing in mind that here m, n, p and q are all integers) – the first pair of coordinates would always be integers, whilst the second pair would always be irrationals.

Another option so far unexplored is using a linear vector walk. If you choose a pair of parallel vectors, then you wouldn't be able to reduce your walk into x- and y- vectors; for example, $\binom{6}{3}$ and $\binom{4}{2}$. These vectors only reduce as far as $\binom{2}{1}$, so all of the reachable points on the vector walk lie along the line $2y = x$, positioned at regular intervals along the line. Coordinates of the form $(2y, y)$ and $(2m, n)$ can clearly coincide at $y = m = n$, so this line would not do for our purposes. Using similar arguments to before, we find that the only lines that could never coincide with the vector walk b would have to have an irrational gradient. This would be true since for every integer x-coordinate you would have an irrational y-coordinate.

Finding a vector walk that visits all integer coordinates.

This should be quite easy now. Although there are plenty of trivial examples I could offer up, I have chosen

$$d_1 = \binom{3}{2}, d_2 = \binom{2}{1},$$

since

$$d_x = (2d_1 - 3d_2) = \binom{0}{1}, d_y = 2d_2 - d_1 = \binom{1}{0},$$

And so I can reach and coordinates of the form (m, n).