Farey neighbours

It must be proved that

$$\frac{b}{d} < \frac{a+b}{c+d} < \frac{a}{c}$$

This can be split into two inequalities

$$\frac{b}{d} < \frac{a+b}{c+d}$$

and

$$\frac{a+b}{c+d} < \frac{a}{c}$$

To prove the first inequality:

$$\frac{b}{d} < \frac{a+b}{c+d}$$

$$\rightarrow b(c+d) < d(a+b)$$

$$\rightarrow bc + bd < ad + bd$$

$$\rightarrow bc < ad$$

$$\rightarrow \frac{b}{d} < \frac{a}{c}$$

The last inequality is true, therefore, the inverse of all the above steps can be carried out to obtain that $\frac{b}{d} < \frac{a+b}{c+d}$. A similar process can be carried out with the second inequality to confirm its truth

Let $\{F_i : i \epsilon \mathbb{N}\}$ be the family of Farey sequences. Let $f_n : \mathbb{N} \rightarrow F_n$ such that $f_n(k)$ is the k^{th} Farey number in F_n $(k \epsilon \mathbb{N})$.

Let $f_n(k) = \frac{b}{d}$ and $f_n(k+1) = \frac{a}{c}$, such that $c + d \leq n + 1$. The aim is to show that $ad - bc = 1$, which is the pattern observed for the Farey sequences I worked out.

Let it be assumed that $ad - bc = 1$ is true, i.e, the pattern holds for $f_n(k)$ and $f_n(k+1)$. Then, it needs to be shown that the pattern holds for $f_{n+1}(k+1)$ and $f_{n+1}(k+2)$ (The pattern holds for $\frac{b}{d}$ and $\frac{a}{c}$ when $c + d > n + 1$, since, in this case, $f_n(k) = f_{n+1}(k+1)$, and $f_n(k+1) = f_{n+1}(k+2)$, so if $f_n(k)$ and $f_n(k+1)$ follow the pattern, then $f_{n+1}(k+1)$ and $f_{n+1}(k+2)$ will also follow the pattern; this is why the above restrictions were placed on $\frac{b}{d}$ and $\frac{a}{c}$)

$$f_{n+1}(k+1) = \frac{a+b}{c+d}$$

and

$$f_{n+1}(k+2) = \frac{a}{c}$$

To prove that the pattern holds for $f_{n+1}(k+1)$ and $f_{n+1}(k+2)$ if it is assumed that the pattern holds for $f_n(k)$ and $f_n(k+1)$, the following calculation needs to be done:

$$a(c+d) - c(a+b)$$

$$= ac + ad - ac - bc$$

$$= ad - bc$$

$$= 1$$

Therefore, it is true that, if the pattern holds for $f_n(k)$ and $f_n(k+1)$, the pattern also holds for $f_{n+1}(k+1)$ and $f_{n+1}(k+2)$

$f_1(1) = \frac{0}{1}$ and $f_1(2) = \frac{1}{1}$. Since $(1)(1) - (0)(1) = 1$, the pattern holds for $f_1(1)$ and $f_1(2)$, so it holds for $f_2(2)$ and $f_2(3)$ and, more generally $f_n(n)$ and $f_n(n+1)$, so we have shown that the pattern holds **across** the Farey sequences

Now, we need to prove that the pattern holds **within** the Farey sequences. Three consecutive numbers in F_n are $f_n(k) = \frac{b}{d}$, $f_n(k+1) = \frac{a+b}{c+d}$ and $f_n(k+2) = \frac{a}{c}$

Let it be assumed that the pattern holds for $f_n(k)$ and $f_n(k+1)$. It must then be proved that the pattern holds for $f_n(k+1)$ and $f_n(k+2)$.

For $f_n(k)$ and $f_n(k+1)$, it is assumed that

$$d(a+b) - b(c+d) = 1$$

$$\rightarrow ad + bd - bc - bd = 1$$

$$\rightarrow ad - bc = 1$$

is true

For $f_n(k+1)$ and $f_n(k+2)$, the following calculation must be done

$$a(c+d) - c(a+b)$$

$$= ac + ad - ac - bc$$

$$= ad - bc = 1$$

,which is true if the pattern holds for $f_n(k)$ and $f_n(k+1)$, so, if the pattern holds for $f_n(k)$ and $f_n(k+1)$, then it also holds for $f_n(k+1)$ and $f_n(k+2)$. Similarly, it can be shown that, if a pattern holds for $f_n(k+2)$ and $f_n(k+1)$, then it holds for $f_n(k+1)$ and $f_n(k)$. Since it has been shown that the pattern is true for $f_n(n)$ and $f_n(n+1)$, the pattern is now true for each pair of consecutive rational numbers in every Farey sequence.