Erratic Quadratic

Pre-requisites:

- C2 level calculus
- FP1 Newton-Raphson iteration formula, matrices
- S1 regression line

To this problem I found 2 solutions, 1 of which is trivial but managed to pass through all the points and is a (kind of) quadratic. First I will explain the non-trivial solution.

Solution 1:

The first thing I noticed about the points is that if a function was to pass through all the points it would have to be of the form f(x) = f(-x) as f(1) = f(-1) = 0, f(2) = f(-2) = 4 etc. This must then mean that f(x) = ax^2 + b, where a and b are some constants we have to find, and (by thinking about the graph of this function) a>0 and b<0. Another way of realising this is that the roots of this equation are the same on both sides of (x=+1 and x=-1) and so the equation is of the form f(x) = a(x-c)(x+c) = ax^2 – ac^2 = ax^2 + b, b=-ac^2. From S1 we were taught how to work out the line of regression to make a linear equation that fits points, and I wondered if this was possible for a quadratic as well. Then I remembered that the idea behind the line is to reduce the sum of the distances from the line to the points. Using this as a basis and a bit of pythag I found a starting point for working out the values of a and b:

$$r_k = \sqrt{(x - x_k)^2 + (y - y_k)^2}$$
$$\therefore r_k^2 = (x - x_k)^2 + (y - y_k)^2$$

Where r_k is the distance between the function at point (x,y) and one of the 6 points (x_k, y_k), where (x_k, y_k) is the k-th point we have been given. We can then substitute y = f(x) = ax^2+b and get:

$$r_k^2 = (x - x_k)^2 + (ax^2 + b - y_k)^2$$

Next I substituted x_k in for x as this will be very close to the function if it is a quadratic line of best fit:

$$r_k^2 = (ax_k^2 + b - y_k)^2$$

This can’t be square-rooted as it needs to be positive or the sum will cancel. Now for the trickiest bit – a new idea. If you imagine that for each r, x and y are constant then r is a function of a and b only. So in equations that’s:

$$r_k^2 = g(a, b)$$
$$g(a, b) = (ax_k^2 + b - y_k)^2$$

Now we want this function as small as possible so we need to look for a minimum point of this equation. To find the lowest point I tried to extend how you find the minimum of a 1 variable function to a 2 variable function. With 1 variable you have to differentiate the equation with respect to the variable and then set it to 0, so I thought why not do this with both of the variables and get 2 equations set to 0, but then you have the problem of differentiating a with respect to b. The da/db problem I solved by thinking of it from a physics point of view that when you have 2 independent variables and you change 1, the other wont change – if you have a light and you’re measuring the light intensity from a distance then the result will change if you change the distance from the light or if you change the initial brightness of the light, but the initial brightness wont change if you change the distance – the same way that 5 wont change as you change x so d5/dx=0 and so da/db=0. Using this result you can differentiate g with respect to a and b to find the minimum of the function g (if you want to check it then expand it out yourself):

$$\frac{dg(a,b)}{da} = 2ax_k^4 + 2bx_k^2 - 2y_kx_k^2$$

$$\frac{dg(a,b)}{db} = 2ax_k^2 + 2b - 2y_k$$

These equations can be set to 0 and simplified to give:

$$ax_k^4 + bx_k^2 = y_kx_k^2$$

$$ax_k^2 + b = y_k$$

That's the complicated bit over, unless you don't like matrices. You have probably noticed that the top equation can be divided by x_k^2 to simplify it, but remember that we have to sum it and the 2s would have cancelled in the sum but the (sum of x)^2 is not always equal to the sum of x^2 and by the same reasoning these can't cancel. In equations that's:

$$\sum x^2 \neq \left(\sum x\right)^2$$

So now all we need to do is invent some new constants:

$$\sum_1^n x_k^4 = \lambda,\ \sum_1^n x_k^2 = \mu,\ \sum_1^n 1 = n,\ \sum_1^n y_kx_k^2 = \alpha,\ \sum_1^n y_k = \beta,$$

And then sum the original 2 equations:

$$\sum_1^n \left(ax_k^4 + bx_k^2\right) = \sum_1^n \left(y_kx_k^2\right)$$

Equation 1: $$a\sum_1^n x_k^4 + b\sum_1^n x_k^2 = \sum_1^n y_kx_k^2$$

$$\sum_1^n \left(ax_k^2 + b\right) = \sum_1^n \left(y_k\right)$$

Equation 2: $$a\sum_1^n x_k^2 + b\sum_1^n 1 = \sum_1^n y_k$$

And then substitute in our new constants to give:

$$a\lambda + b\mu = \alpha$$

$$a\mu + bn = \beta$$

Which you will realise are simultaneous equations. To solve this you can use matrices:

$$\begin{pmatrix}\lambda & \mu\\ \mu & n\end{pmatrix}\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{pmatrix}a\\ b\end{pmatrix} = \begin{pmatrix}\lambda & \mu\\ \mu & n\end{pmatrix}^{-1}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{pmatrix}a\\ b\end{pmatrix} = \frac{1}{\lambda n - \mu^2}\begin{pmatrix}n\alpha - \mu\beta\\ \lambda\beta - \mu\alpha\end{pmatrix}$$

And therefore:

$$a = \frac{n\alpha - \mu\beta}{\lambda n - \mu^2},\ b = \frac{\lambda\beta - \mu\alpha}{\lambda n - \mu^2}$$

These formulas can now be used for any set of points that are symmetric through the x-axis.
Now we must work out the values of the new constants so we can find the value of a and b. These values are (without showing the working):
$\lambda = 196, \mu = 28, n = 6, \alpha = 320, \beta = 40$
Which you can verify if you want. If you plug this into the formulas for a and b you get a=100/49 and b=-20/7. This fits our initial idea that a>0 and b<0 so it should work. So our final function is f(x) = (10x/7)^2 – 20/7.

$$f(x) = \frac{100x^2}{49} - \frac{20}{7}$$

How to find the smallest distance between f(x) and the points given:
The smallest distance will lie along the normal of the line at a point where the normal also crosses the point given. In other words, a point on the line where the normal to that part of the line crosses the given point is the closest part of the line.
So that means that first we must work out the derivative of f(x):

$$f(x) = \frac{100x^2}{49} - \frac{20}{7}$$
$$f'(x) = \frac{200x}{49}$$

We can then include this in the formula for a straight line to work out the normal at each point and denote this as g(x):

$$y - y_0 = m(x - x_0)$$
$$y = \frac{-(x - x_0)}{f'(x_0)} + f(x_0)$$
$$g(x) = \frac{49(x_0 - x)}{200x_0} + \left(\frac{100x_0^2}{49} - \frac{20}{7}\right)$$
$$g(x) = \frac{100x_0^2}{49} - \frac{49x}{200x_0} - \frac{3657}{1400}$$

Now if (x_k, y_k) lies on the line g(x) then g(x_k) = y_k and so:

$$y_k = \frac{100x_0^2}{49} - \frac{49x_k}{200x_0} - \frac{3657}{1400}$$

This can be arranged into a cubic to be solved for x_0:

$$ax_0^3 + (b - y_k)x_0 + cx_k = 0$$
$$a = \frac{100}{49}, b = \frac{-3657}{1400}, c = \frac{-49}{200}$$

Now I don't know how to solve this but I do know how to use the Newton-Raphson iteration formula to solve it. x_0 will now be replaced by α_n and iteratively found so that α_∞ = x_0. The iteration formula is:

$$\alpha_{n+1} = \alpha_n - \frac{a\alpha_n^3 + (b - y_k)\alpha_n + cx_k}{3a\alpha_n^2 + b - y_k}$$
$$\alpha_0 = x_k$$

I chose x_k as a starting point as this will be very close to the x-coordinate anyway.
With this you can find the x-coordinate of the closest point to (x_k, y_k). From this you can then work out the distances by pythag again:

$$r_k = \sqrt{(\alpha_\infty - x_k)^2 + (f(\alpha_\infty) - y_k)^2}$$
$$r_k = \sqrt{(\alpha_\infty - x_k)^2 + \left(\left(\frac{10\alpha_\infty}{7}\right)^2 - \frac{20}{7} - y_k\right)^2}$$
Using this formula you can work out the smallest distances. Here is a table to sum up all of the properties of each point:

k	x_k	y_k	$f(x_k)$	α_∞	$f(\alpha_\infty)$	$\|r_k\|$
1	1	0	−0.8163	1.1756	−0.0366	0.1794
2	2	4	5.3061	1.8360	4.0219	0.1655
3	3	16	15.5102	3.0395	15.9968	0.0396
4	−1	0	−0.8163	−1.1756	−0.0366	0.1794
5	−2	4	5.3061	−1.8360	4.0219	0.1655
6	−3	16	15.5102	−3.0395	15.9968	0.0396

As you can see it may not be close for each predicted y-coordinate, but it does get very close for the iterated points and never exceeds being 0.2 units away from any of the points given.

Solution 2 – the trivial one:
As well as noticing that f(x) = f(-x) I noticed another interesting property of this function (assuming the function goes through all the points). Imagine that it was a normal x^2 graph when x>0, then the x values would be 0, 2, 4. If you match these up with the ones we are given and call the normal ones u and the given ones x then you'll notice a basic pattern emerge : x->u, 1->0, 2->2, 3->4, … , x->2x-2. Therefore u=2(x-1), x>0. If you do this with x<0 then the pattern becomes u=2(x+1), x<0. This means that u can be written as a function of x (ie u=g(x)):
$$g(x) = 2\begin{cases} x-1, & x>0 \\ x+1, & x<0 \end{cases}$$
This function returns the value of u, which we can then square to get the actual value of the function. Or mathematically:
$$f(x) = g(x)^2$$
We can also simplify the function g using the sgn function which returns the sign of the variable. In other words sgn(+5)=+1, sgn(-123)=-1, sgn(0)=0. Also note that sgn(x)^2 = 1, unless x=0. This simplifies g(x) to:
$$g(x) = 2(x - sgn(x))$$
This can then be substituted into f(x) and simplified to give:
$$f(x) = 4(x - sgn(x))^2$$
$$f(x) = 4x^2 - 8|x| + 4$$
$$f(x) = 4x^2 - 8\sqrt{x^2} + 4$$
This formula works, but breaks down when asked what f(0) is, and its not really a proper quadratic, but other than that it can get all the right results and is 0 units away from every point.