Solution to the Magic Bag problem

Let n = the number of balls in the bag.
Let b = the number of black balls and w = the number of white balls.

Clearly $n = w + b$, so we can replace w with $n - b$.
Without loss of generality we will assume that $w \geq b$.

P(both black *or* both white) $= \dfrac{b}{n} \cdot \dfrac{b-1}{n-1} + \dfrac{n-b}{n} \cdot \dfrac{n-b-1}{n-1}$

which simplifies to $\dfrac{2b^2 - 2bn - n + n^2}{n(n-1)}$

Equating this to ½ gives the equation: $n^2 - n = 4b^2 - 4bn - 2n + 2n^2$

and this in turn simplifies to: $n = (n - 2b)^2$

$\therefore \quad n = (w - b)^2$ since $w = n - b$

So n must be a square number, and given the context $n > 1$, so $w \neq b$.

Now, consider T_k, the k^{th} triangular number: $T_k = \sum_{r=1}^{k} r = \dfrac{k(k+1)}{2}$

$\therefore \quad 2T_k = k^2 + k$
$\therefore \quad k^2 = 2T_k - k$

So every square number can be written as a function of its corresponding triangular number and their index.

Thus since n is a square number, $\therefore \quad n = 2T_{\sqrt{n}} - \sqrt{n}$

Rearranging this gives $\tfrac{1}{2}(n + \sqrt{n}) = T_{\sqrt{n}}$

$\therefore \quad \tfrac{1}{2}(n + w - b) = T_{\sqrt{n}}$ since $n = (w - b)^2$

$\therefore \quad \tfrac{1}{2}(2w) = T_{\sqrt{n}}$ since $w = n - b$

giving the result that $w = T_{\sqrt{n}}$

So in order for the probability to be ½, the total number of balls must be a square number and the number of white balls must be the corresponding triangular number.
If we take n, the total number of balls, as k^2, then $w = \tfrac{1}{2}k(k+1)$

This establishes the number of black balls as $b = k^2 - \tfrac{1}{2}k(k+1) = \tfrac{1}{2}k(k-1)$, i.e. the previous triangular number.

In conclusion, the numbers of white and black balls must be consecutive triangular numbers in order for the given probability to equal ½.

We have proved the necessity of this condition. It is a relatively simple matter to prove sufficiency:

Given $\frac{1}{2}k(k+1)$ white balls and $\frac{1}{2}k(k-1)$ black balls in a bag, the probability of two randomly selected balls being the same colour is:

$$\frac{\frac{1}{2}k(k+1)}{k^2}\cdot\frac{\frac{1}{2}k(k+1)-1}{k^2-1}+\frac{\frac{1}{2}k(k-1)}{k^2}\cdot\frac{\frac{1}{2}k(k-1)-1}{k^2-1}$$

which (eventually) simplifies to $\frac{1}{2}$.

This wraps it up nicely: the scenario described occurs with probability ½ if and only if the numbers of black and white balls in the magic bag are consecutive triangular numbers.