NRICH "Mechanical Mindgames", PART 3, Elliott Gordon

Two spheres of solid lead of radius 1m are in deep space stationary relative to each other and a fixed origin with a distance of 3m between their centres. If we ignore all gravitational effects other than those due to the two spheres, estimate how fast they will be moving when they strike. You might also try to estimate how long it will take before they collide. How do the results change for 1cm lead spheres with a distance of 3cm between their centres?
When a particle falls under the action of gravity, the acceleration is given by Newton's 2nd law $F=ma$ *and the magnitude of the force by Newton's law of gravity* $F=-GmM/r^2$

In order to find the force acting on each sphere, one must calculate their mass. Since their volume is given, finding a value of their density will suffice this purpose. Wikipedia says that lead has a density of $11.34gcm^{-3}$. Turning this into SI units:

$$d = \frac{11.34g}{1cm^3} \times \frac{1kg}{1000g} \times \frac{10^6 cm^3}{1m^3} = 11340kgm^{-3}$$

The volume of each sphere is given by the formula (using r=1):

$$V = \frac{4}{3}\pi r^3$$

So the mass of each sphere is given by:

$$m = Vd = 4.1888 \times 11340 = 47501kg$$

Finding the net force on each sphere is complex, since each atom in one sphere receives different forces from different atoms in the other. The net force on the sphere would itself be the sum of all of the forces on each individual atom, overcomplicating the analysis. Since the question asks to *estimate* the speed, each sphere may be treated as a point mass (located in its centre) to simplify the situation. That said, a possible approach to solve the problem is using energy considerations:

Let the fixed origin be in the midpoint between the two spheres, considered as point masses A and B. Since they receive equal and opposite forces, and are placed symmetrically with respect to the origin, they will move symmetrically with respect to it. Let x be the distance from the origin to each of the point masses at a moment in time. At that moment, the force on each mass will be:

$$F = -G\frac{Mm}{r^2} = -G\frac{m^2}{(2x)^2} = -G\frac{m^2}{4x^2}$$

Since the energy transferred to an object by a force F acting along a distance d is given by $W = Fd$, the energy transferred to each mass by the gravitational force is equivalent to the area enclosed by the graph of F against x between the two values of x that determine the traversed distance d. Since the masses start 1.5 meters from the origin, and touch at the origin (with their centres each at a distance of 1 meter from the origin), the two values of x will be 1.5 and 1. The value of this area can now be evaluated by the integral:

$$W = \int_{1}^{1.5} -G\frac{m^2}{4x^2}dx = [G\frac{m^2}{4x}]_1^{1.5}$$

Substituting in the values of the constant $G = 6.67 \times 10^{-11}$ and the mass $m = 47500$ to evaluate the integral:

$$W = \frac{Gm^2}{4(1.5)} - \frac{Gm^2}{4(1)} = 0.025082 - 0.037623 = -0.012541J$$

That is the amount of gravitational potential energy gained by each sphere from the initial situation until the time of contact. All this energy is turned into kinetic energy, so:

$$W = \frac{1}{2}mv^2 = 0.012541$$

Making v the subject to compute a numerical answer:

$$v = \sqrt{2\frac{W}{m}} = 7.27 \times 10^{-4}$$

One can thus estimate that the spheres will strike with a speed of $7.27 \times 10^{-4} ms^{-1}$ each.

There is another approach to the problem that doesn't involve energies:

As established previously, placing the origin in the midpoint of the spheres with a distance x from it to each point mass results in a force on either:

$$F = -G\frac{m^2}{4x^2}$$

Dividing both sides by m and applying Newton's third law $F = ma$ gives an acceleration of:

$$\frac{F}{m} = \mathrm{a} = -G\frac{m}{4x^2}$$

But the definition of acceleration is:

$$a = \frac{dv}{dt}$$

Applying the chain rule of differentiation:

$$a = \frac{dv}{dx} \times \frac{dx}{dt}$$

But the definition of velocity is:

$$v = \frac{dx}{dt}$$

Substituting this into the first equation:

$$a = v\frac{dv}{dx}$$

Substituting in the initial expression for a:

$$v\frac{dv}{dx} = -G\frac{m}{4x^2}$$

Integrating both sides of the equation with respect to x gives:

$$\int v\frac{dv}{dx}dx = \int -G\frac{m}{4x^2}dx$$

The dx "cancels out" on the left hand side, and the integral to the right can be found:

$$\int vdv = G\frac{m}{4x} + \mathrm{c}$$

The left hand side can now be integrated:

$$\frac{1}{2}v^2 = G\frac{m}{4x} + \mathrm{c}$$

Solving the differential equation has given an equation with v in terms of x, which will give the required solution. First the value of c must be found in order to complete the equation. When x=1.5 the spheres are in the initial position, and therefore stationary, so v=0. Substituting these two values into the equation gives the correct value of c:

$$0 = G\frac{m}{4(1.5)} + \mathrm{c}$$

Solving and computing the result:

$$c = -G\frac{m}{6} = -5.2804 \times 10^{-7}$$

Now, knowing the fact that the spheres touch at x=1, and using this value of c, the corresponding velocity may be found:

$$\frac{1}{2}v^2 = G\frac{m}{4} + \mathrm{c}$$

Making v the subject and computing the result:

$$v = \sqrt{2(G\frac{m}{4} + \mathrm{c})} = 7.27 \times 10^{-4}$$

Again, by this different method it is also possible to estimate that the spheres will have a speed of $7.27 \times 10^{-4} ms^{-1}$ at the moment of impact.

To estimate the time taken for the spheres to collide, one could use a midpoint value of the speeds that the sphere takes along its trail (i.e. halving the maximum speed of $7.27 \times 10^{-4} ms^{-1}$ gives a midpoint speed of $3.64 \times 10^{-4} ms^{-1}$). Knowing that the total distance travelled by the sphere is 0.5 meters, the time is given by:

$$t = \frac{d}{v} = \frac{0.5}{3.64 \times 10^{-4}} = 1373 \approx 1000s$$

Thus, a possible estimate of the time taken for the spheres to collide is 1000 seconds.

(To calculate the time taken for the spheres to collide, one could use the solution to the previous differential equation:

$$\frac{1}{2}v^2 = G\frac{m}{4x} + \mathrm{c}$$

Then make the inverse of v the subject of the equation and integrate with respect to x (this gives the time in function of the position x). Solving this differential equation should give an accurate value of the time taken)

For spheres of 1cm with 3cm between their centres:
The mass of each sphere is:

$$m = Vd = \frac{4}{3}\pi \times (10^{-2})^3 \times 11340 = 0.047501kg$$

Repeating the analysis for the new dimensions gives the same differential equation with the previous solution:

$$\frac{1}{2}v^2 = G\frac{m}{4x} + \mathrm{c}$$

But with a different value of c (calculated for x=1.5cm, v=0)

$$c = -G\frac{m}{4(1.5 \times 10^{-2})} = -5.2804 \times 10^{-11}$$

Using this value and x=1cm (when the spheres touch) in the equation gives the corresponding v:

$$v = \sqrt{2(G\frac{m}{4(10^{-2})} + \mathrm{c})} = 7.26 \times 10^{-6}$$

And, again, using a midpoint value for the speed of $3.64 \times 10^{-6} ms^{-1}$ allows an estimation of the time taken:

$$t = \frac{d}{v} = \frac{0.5 \times 10^{-2}}{3.64 \times 10^{-6}} = 1373 \approx 1000$$

It can thus be estimated that the spheres touch after $1000s$ with a speed of $7.26 \times 10^{-6} ms^{-1}$. This is the same amount of time taken as the larger spheres, and a speed 100 times smaller. This is because the masses vary bya factor of 10^6 and the distances vary by a factor of 10^2, so for the small spheres the acceleration decreases by a factor of 10^6 and increases twice by a factor of 10^2: the acceleration is 100 times smaller, and therefore the final velocity is 100 times smaller as well (over a distance 100 times smaller this results in the same time taken).

When a particle falls under the action of gravity, the acceleration is given by Newton's 2nd lawF=maand the magnitude of the force by Newton's law of gravityF=−GmM/r². Experimentation indicates that the numerical value of the littlem
in each equation is identical, and the principle of equivalence (which appears to be true) asserts that they are mathematically identical. Think about this; it is good to understand. Does it surprise you?

By equating the two F's and cancelling the m's one obtains an expression for a:

$$a = -G\frac{m}{r^2}$$

The main implication of the two m's being the same is thatthe acceleration experienced by a mass under gravity depends on the mass of the object it is attracted to and the distance between them, but does not depend on the mass of the object itself. We therefore speak of the *acceleration* of gravity rather than the *force* of gravity when we solve problems concerned with motion under this force. This can, at first, seem surprising and counterintuitive: the classic example of a leaf or feather being dropped simultaneously to a stone (the stone falls faster) seems to contradict the theory. One could think that the smaller mass of the feather reduces the gravitational pull acting on it, whereas the greater mass of the stone gives it a greater acceleration. However, it is actually the greater amount of air resistance on the feather that significantly reduces its acceleration due to its small mass. If both the stone and the feather (or any two objects) were dropped in a vacuum, the two would experience the exact same motion regardless of their masses, since there would be no air resistance.

A certain mathematical particle is defined to have speed 1 for irrational values of time and speed 0 for rational values of time. Would it make sense for this particle to move?
[Note from NRICH: it might also be possible to argue that the particle does move if notions of speed and integration are redefined. This is a persuasive argument that the particle would remain fixed based on standard notions of speed and integration]

Such a particle would not move because:
The distance moved by a particle with a speed v during a time interval Δt is given by:

$$\Delta x = v\Delta t$$

It is obvious from the formula that in order for the particle to move (for $\Delta x > 0$), both v and Δt must be greater than 0: $v, \Delta t > 0$.
In the case of the particle under consideration, $v = 0$ when time is rational, so, for these time values, the particle will not move. For irrational time values, $v = 1$, so the particle will only move if there exists a time interval $\Delta t = t_2 - t_1 > 0$ in which the particle can sustain this speed. In other words, in order for the particle to move, there must exist two real numbers, α andβ, where$\alpha - \beta > 0$, such that the interval $]\alpha, \beta[$ only contains irrational numbers (i.e. contains no rationals), in symbols: if $x \in]\alpha, \beta[$, then $x \in \mathbb{R} \setminus \mathbb{Q}$. However, no two such α and βexist.

Proof:

Let α and βbe two real numbers where $\alpha > \beta$ (and therefore$\alpha - \beta > 0$)

If the interval $]\alpha, \beta[$ encloses an integer, it also encloses a rational (because integers are also rationals).

If the interval $]\alpha, \beta[$ does not enclose an integer, then the numbers α and βhave the same integral part (they only differ in their decimal part), and may be written as a string of digits with an infinite decimal expansion:

$$\alpha = N_m \dots N_2 N_1 . a_1 a_2 a_3 \dots$$

$$\beta = N_m \dots N_2 N_1 . b_1 b_2 b_3 \dots$$

Since $\alpha \neq \beta$, there must be some point along the decimal expansion of α and β where their digits are different. Let the first different decimals be $a_n \neq b_n$ (in the special case that $a_n = 1 + b_n$ and that after the first different decimal, α has a string of 0's in its expansion whilst β has a string of 9's, take $a_n \neq b_n$ to be instead the second set of two different decimals other than 0 and 9 in the expansion, which must exist because otherwise $\alpha = \beta$). Because $\alpha > \beta$, then necessarily $a_n > b_n$.

Now, let a_m,(where $m > n$) be the second non-zero decimal after a_n in the expansion of α. Then, the number

$r = \alpha - 0.r_1 r_2 \dots r_{m-1} a_m a_{m+1} \dots = \alpha - 0.00 \dots 0 a_m a_{m+1} \dots; (r_1, r_2, \dots r_{m-1} = 0)$

which can be rewritten as

$$r = N_m \dots N_2 N_1 . a_1 a_2 a_3 \dots a_{m-1}$$

isnecessarily contained within the interval $]\alpha, \beta[$ and is rational since it has a finite amount of decimals.

In the case that a_m does not exist (i.e. every decimal after a_n is 0), then let b_m be the second non-9 digit after b_n in the expansion of β. Then, the number

$r = \beta + 0.r_1 r_2 \dots r_{m-1} c_m c_{m+1} \dots$; where$r_1, r_2, \dots r_{m-1} = 0$ and where c_m is defined to be the number such that $c_m + b_m = 9$

can also be written as

$$r = N_m \dots N_2 N_1 . b_1 b_2 \dots b_{m-1} 999 \dots$$

Again, the number r is necessarily contained within the interval $]\alpha, \beta[$ and is rational since it ends in a periodic string of 9's.

Thus, in every case, there is a rational number contained within any interval of real numbers $]\alpha, \beta[$. Q.E.D.

Therefore, there exists no interval of real numbers $]\alpha, \beta[$ where $\alpha - \beta > 0$, that only includes irrational numbers, so there exists no time interval $\Delta t > 0$ along which the particle has a positive speed, so the particle does not move.