Angle of Shot Part I

Here is the formula to calculate the distance, D, a projectile will travel when thrown with velocity v, from an initial height y_0 and an angle of trajectory θ with acceleration due to gravity of g:

$$D = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{(v\sin\theta)^2 + 2gy_0}\right)$$

To find the value of θ that maximises D when v, g and y_0 are constant, we must differentiate with respect to theta. Using a combination of the product rule and the chain rule:

$$\frac{dD}{d\theta} = \frac{-v\sin\theta}{g}\left(v\sin\theta + \sqrt{(v\sin\theta)^2 + 2gy_0}\right) + \frac{v\cos\theta}{g}\left(v\cos\theta + \frac{1}{2}(v^2\sin^2\theta + 2gy_0)^{-\frac{1}{2}}(2v^2\sin\theta\cos\theta)\right)$$

For the purposes of simplification, suppose that:

$$\alpha = 2gy_0$$

Then:

$$\frac{dD}{d\theta} = \frac{-v\sin\theta}{g}\left(v\sin\theta + \sqrt{(v\sin\theta)^2 + \alpha}\right) + \frac{v\cos\theta}{g}\left(v\cos\theta + \frac{1}{2}(v^2\sin^2\theta + \alpha)^{-\frac{1}{2}}(2v^2\sin\theta\cos\theta)\right)$$

$$\frac{dD}{d\theta} = \frac{-v\sin\theta}{g}\left(v\sin\theta + \sqrt{(v\sin\theta)^2 + \alpha}\right) + \frac{v\cos\theta}{g}\left(v\cos\theta + (v^2\sin^2\theta + \alpha)^{-\frac{1}{2}}(v^2\sin\theta\cos\theta)\right)$$

$$\frac{dD}{d\theta} = \frac{1}{g}\left(-v\sin\theta\left(v\sin\theta + \sqrt{(v\sin\theta)^2 + \alpha}\right) + v\cos\theta\left(v\cos\theta + (v^2\sin^2\theta + \alpha)^{-\frac{1}{2}}(v^2\sin\theta\cos\theta)\right)\right)$$

$$\frac{dD}{d\theta} = \frac{1}{g}\left(-v^2\sin^2\theta - v\sin\theta\sqrt{v^2\sin^2\theta + \alpha} + v^2\cos^2\theta + \frac{v\cos\theta(v^2\sin\theta\cos\theta)}{\sqrt{v^2\sin^2\theta + \alpha}}\right)$$

Once again for simplification, suppose that:

$$c = \frac{\alpha}{v^2}$$

Then:

$$\alpha = cv^2$$

Therefore:

$$\frac{dD}{d\theta} = \frac{1}{g}\left(-v^2\sin^2\theta - v\sin\theta\sqrt{v^2[(\sin]^2\theta + c)} + v^2\cos^2\theta + \frac{v\cos\theta(v^2\sin\theta\cos\theta)}{\sqrt{v^2(\sin^2\theta + c)}}\right)$$

$$\frac{dD}{d\theta} = \frac{1}{g}\left(-v^2\sin^2\theta - v^2\sin\theta\sqrt{\sin^2\theta + c} + v^2\cos^2\theta + \frac{v\cos\theta(v^2\sin\theta\cos\theta)}{v\sqrt{\sin^2\theta + c}}\right)$$

$$\frac{dD}{d\theta} = \frac{1}{g}\left(-v^2\sin^2\theta - v^2\sin\theta\sqrt{\sin^2\theta + c} + v^2\cos^2\theta + \frac{v^2\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + c}}\right)$$

$$\frac{dD}{d\theta} = \frac{v^2}{g}\left(-\sin^2\theta - \sin\theta\sqrt{\sin^2\theta + c} + \cos^2\theta + \frac{\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + c}}\right)$$

When D is at a maximum, the derivative of D with respect to θ is equal to zero. Therefore:

$$\frac{v^2}{g}\left(-sin^2\theta - sin\theta\sqrt{sin^2\theta + c} + cos^2\theta + \frac{sin\theta cos^2\theta}{\sqrt{sin^2\theta + c}}\right) = 0$$

$$-sin^2\theta - sin\theta\sqrt{sin^2\theta + c} + cos^2\theta + \frac{sin\theta cos^2\theta}{\sqrt{sin^2\theta + c}} = 0$$

$$cos^2\theta + \frac{sin\theta cos^2\theta}{\sqrt{sin^2\theta + c}} - sin^2\theta - sin\theta\sqrt{sin^2\theta + c} = 0$$

Suppose, for the purposes of simplification, that:

$$k = cos^2\theta$$

Then:

$$1 - k = sin^2\theta$$

Therefore with some manipulation and a messy expansion the equation can be simplified:

$$k + \frac{k\sqrt{1-k}}{\sqrt{1-k+c}} - (1-k) - (\sqrt{1-k})(\sqrt{1-k+c}) = 0$$

$$2k - 1 + \frac{k\sqrt{1-k}}{\sqrt{1-k+c}} - \frac{(\sqrt{1-k})(\sqrt{1-k+c})^2}{\sqrt{1-k+c}} = 0$$

$$2k - 1 + \frac{k\sqrt{1-k} - (1-k+c)\sqrt{1-k}}{\sqrt{1-k+c}} = 0$$

$$2k - 1 + \frac{(k-1+k-c)\sqrt{1-k}}{\sqrt{1-k+c}} = 0$$

$$\frac{(2k-1-c)\sqrt{1-k}}{\sqrt{1-k+c}} = 1 - 2k$$

$$\frac{(2k-1-c)^2(1-k)}{(1-k+c)} = (1-2k)^2$$

$$(4k^2 - 2k - 2kc - 2k + 1 + c - 2kc + c + c^2)(1-k) = (4k^2 - 4k + 1)(1-k+c)$$

$$(4k^2 + 1 + c^2 - 4k - 4kc + 2c)(1-k) = (4k^2 - 4k + 1)(1-k+c)$$

Expanding gives:

$$4k^2 + 1 + c^2 - 4k - 4kc + 2c - 4k^3 - k - kc^2 + 4k^2 + 4k^2c - 2kc = 4k^2 - 4k + 1 - 4k^3 + 4k^2 - k + 4k^2c - 4kc +$$

Which cancels down into:

$$c^2 + 2c - kc^2 - 2kc = c$$

$$kc^2 + 2kc = c^2 + c$$

$$kc(c+2) = c(c+1)$$

$$k = \frac{c(c+1)}{c(c+2)}$$

$$k = \frac{c+1}{c+2}$$

But recall what we let k and c represent:

$$k = cos^2\theta$$

$$c = \frac{\alpha}{v^2} = \frac{2gy_0}{v^2}$$

Therefore the value of θ which gives the maximum value of D is given by:

$$cos^2\theta = \frac{\left(\frac{2gy_0}{v^2} + 1\right)}{\left(\frac{2gy_0}{v^2} + 2\right)}$$

$$cos^2\theta = \frac{\left(\frac{2gy_0 + v^2}{v^2}\right)}{\left(\frac{2gy_0 + 2v^2}{v^2}\right)}$$

$$cos^2\theta = \frac{2gy_0 + v^2}{2gy_0 + 2v^2}$$

$$\theta_{optimum} = \arccos\left(\sqrt{\frac{2gy_0 + v^2}{2gy_0 + 2v^2}}\right)$$

Now we can consider reasonable assumptions about the nature of the shot to arrive at a reliable estimate for the optimum angle for the shot putter to throw the shot at. Since the shot is being released from headheight, which is likely to be around 6 feet, we can assume that y_0 is around 1.8m. Acceleration due to gravity at the earth's surface is given to be $9.81ms^{-2}$. Now we can work out an estimate for v. Let us assume that the thrower can put 350 N of force into the shot. Let us also assume that they accelerate the shot for around 0.35 seconds. The mass of a standard shot is 7.26kg. Then:

$$F = ma$$

$$350 = 7.26a$$

$$a = \frac{350}{7.26}$$

$$v = at$$

$$v = \frac{350}{7.26}x0.35 = \frac{245}{14.52}$$

Therefore, using our assumptions, an estimate for our optimum angle is:

$$\theta_{optimum} \approx \arccos\left(\sqrt{\frac{2(9.81)(1.8)+\left(\frac{245}{14.52}\right)^2}{2(9.81)(1.8)+2\left(\frac{245}{14.52}\right)^2}}\right)$$

$$\theta_{optimum} \approx 43^{\circ}$$

So our approximate optimum angle to maximise the shot length is somewhere near 43 degrees.