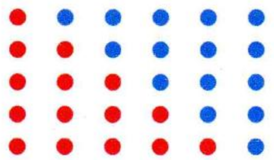
So It's Times!: Stage 2 and 3***

Zach Thompson

This problem was a bit harder than I thought it would be. I had to really think, but in the end, the answer was obvious!

I knew that there would be 8 possible grid combinations, four rotations for each 'face' of the plastic grid. The most obvious, is when one grid is placed on top of the other and the numbers align exactly to give $1^2 + 2^2 + 3^2 + \cdots + 36^2$. Summing this series is exactly the same as the 36th pyramidal number (or square pyramid number). To explain why, I first needed to take a step back, because there is a massive link to triangular numbers.



Triangular numbers are the sum of consecutive numbers.
$T_1 = 1,\ \ T_2 = (1+2),\ \ T_3 = (1+2+3)etc..$

Double any triangle number to make a rectangle or $n(n+1)$

Gauss discovered a fabulous way of summing consecutive numbers. He added two strings of numbers, but reversed the second one, so that the sums would be the same:

	1	2	3	4	5	...	$(n-1)$	n
+	n	$(n-1)$	$(n-2)$	$(n-3)$	$(n-4)$	...	2	1
	$(n+1)$	$(n+1)$	$(n+1)$	$(n+1)$	$(n+1)$	...	$(n+1)$	$(n+1)$

There are n x $(n + 1)$ in total, but Gauss only need to add one string, not two, so the expression is:

$$\frac{n(n+1)}{2}$$

To sum the first 36 numbers, I just needed to calculate the 36th triangular number:

$$T_{36} = \frac{36 \times 37}{2} = 666$$

To sum the first 36 square numbers, I used square pyramidal numbers. Imagine a set of balls with (36x36) at the base, (35x35) placed on top, (34x34) on top, right the way until you finally reach (1x1) at the very top. This is a square pyramid, the formula is:

$$P_n = \frac{n(n+1)(2n+1)}{6} \qquad P_{36} = \frac{36 \times 37 \times 73}{6} = 16{,}206$$

I didn't know if this was within the rules, but I hadn't set out to calculate a figure. I just followed the logic.

Square pyramidal numbers are linked to triangular numbers, because they are actually the sum of consecutive pairs of tetrahedral numbers (triangular pyramid numbers).

I started to create a chart to record my thoughts. I took the biggest and smallest pairs, to give me a good idea of size, but then I had an genius brainwave, which lead to a theory and meant that I could solve the puzzle.

Grid Combination	Three Largest Pairs			Three Smallest Pairs			Initial Thoughts
A	36 x 36	35 x 35	34 x 34	1 x 1	2 x 2	3 x 3	Largest
B	18 x 19	17 x 20	16 x 21	1 x 36	2 x 35	3 x 34	B<A

A) grids align exactly: giving $1^2 + 2^2 + 3^2 + \cdots + 36^2$

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

PREDICTION: HIGHEST GRID COMBINATION

Because (36 x 36), (35 x 35), (34 x 34)... (1 x 1) are the highest value pairings.

(B) grids rotated 180 degrees:

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

36	35	34	33	32	31
30	29	28	27	26	25
24	23	22	21	20	19
18	17	16	15	14	13
12	11	10	9	8	7
6	5	4	3	2	1

PREDICTION:
SMALLEST GRID COMBINATION

Initially, I was thinking that (18 x 19) was smaller than 19^2, and therefore smaller than every subsequent pair in (A). From this, I knew that B would be smaller than A. Next I looked at patterns. There is repetition in the grid combinations, both vertically and horizontally. If you look carefully, then you can see pairs repeat. In fact Q1 and Q3 are the same, whilst Q2 and Q4 are the same.

But then I realised, the answer was staring me in the face, I just had to follow Gauss, but multiply instead. Because I wasn't supposed to work out the total sum, I just used the numbers 1 to 10 to test out a theory I had:

	1	2	3	4	5	6	7	8	9	10
×	10	9	8	7	6	5	4	3	2	1
	10	18	24	28	30	30	28	24	18	10

If I added these products, the total comes to 220. I thought there may be a link to tetrahedral numbers (triangular pyramid numbers). The formula is:

$$Te_n = \frac{n(n+1)(n+2)}{6} \qquad Te_{10} = \frac{10 \times 11 \times 12}{6} = 220$$

I was right! The total is a tetrahedral number! I could now apply this logic to grid B.

I was certain that the square pyramidal number (sum of consecutive squares, grid A) was the largest grid combination, and was convinced that grid B would sum to a tetrahedral number. Because the *nth* tetrahedral number is the sum of the first *n* triangular numbers, this must be the smallest combination. Everything else is likely to lie between the two.

This was my theory, so I then worked out the values of each grid combination to check if I was right.

A) grids align exactly: giving $1^2 + 2^2 + 3^2 + \cdots + 36^2$

Theory: $P_n = \frac{n(n+1)(2n+1)}{6}$ $\qquad P_{36} = \frac{36\times37\times73}{6} = 16{,}206$ so I predict the sum of grid A to be 16,206

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1	4	9	16	25	36	**91**
49	64	81	100	121	144	**559**
169	196	225	256	289	324	**1459**
361	400	441	484	529	576	**2791**
625	676	729	784	841	900	**4555**
961	1024	1089	1156	1225	1296	**6751**
						16206

And I was correct!

(B) grids rotated 180 degrees:

Theory: $Te_n = \frac{n(n+1)(n+2)}{6}$ $\qquad Te_{36} = \frac{36\times37\times38}{6} = 8436$ so I predict the sum of grid B to be 8436

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

36	35	34	33	32	31
30	29	28	27	26	25
24	23	22	21	20	19
18	17	16	15	14	13
12	11	10	9	8	7
6	5	4	3	2	1

36	70	102	132	160	186	**686**
210	232	252	270	286	300	**1550**
312	322	330	336	340	342	**1982**
342	340	336	330	322	312	**1982**
300	286	270	252	232	210	**1550**
186	160	132	102	70	36	**686**
						8436

Again I was correct. I predict that grid A is the largest total and grid B the smallest total. All the following totals will lie between the two i.e. between a square pyramidal (square based pyramid) and a tetrahedral (triangular based pyramid) number.

(C) grids rotated 90 degrees (in both directions): I had a feeling both of these would have the same total.

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

C(i)

6	12	18	24	30	36
5	11	17	23	29	35
4	10	16	22	28	34
3	9	15	21	27	33
2	8	14	20	26	32
1	7	13	19	25	31

C(ii)

31	25	19	13	7	1
32	26	20	14	8	2
33	27	21	15	9	3
34	28	22	16	10	4
35	29	23	17	11	5
36	30	24	18	12	6

I was correct, the totals are the same. This is because the rows in C(i) equal the columns in C(ii).

6	24	54	96	150	216	**546**
35	88	153	230	319	420	**1245**
52	140	240	352	476	612	**1872**
57	180	315	462	621	792	**2427**
50	208	378	560	754	960	**2910**
31	224	429	646	875	1116	**3321**
231	**864**	**1569**	**2346**	**3195**	**4116**	**12321**

31	50	57	52	35	6	**231**
224	208	180	140	88	24	**864**
429	378	315	240	153	54	**1569**
646	560	462	352	230	96	**2346**
875	754	621	476	319	150	**3195**
1116	960	792	612	420	216	**4116**
3321	**2910**	**2427**	**1872**	**1245**	**546**	**12321**

Next I flipped the grids (imagine both grids are on a piece of paper, and then folding between the grids, so they meet).

(D) grids flipped, but no rotation (note a clear mirror image).

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

6	5	4	3	2	1
12	11	10	9	8	7
18	17	16	15	14	13
24	23	22	21	20	19
30	29	28	27	26	25
36	35	34	33	32	31

6	10	12	12	10	6	
84	88	90	90	88	84	
234	238	240	240	238	234	
456	460	462	462	460	456	
750	754	756	756	754	750	
1116	1120	1122	1122	1120	1116	
2646	**2670**	**2682**	**2682**	**2670**	**2646**	**15996**

Staying with a flipped plastic grid, I now applied rotation:

(E) grids flipped, 180 degree rotation: (note another clear mirror image)

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

31	32	33	34	35	36
25	26	27	28	29	30
19	20	21	22	23	24
13	14	15	16	17	18
7	8	9	10	11	12
1	2	3	4	5	6

31	64	99	136	175	216	**721**
175	208	243	280	319	360	**1585**
247	280	315	352	391	432	**2017**
247	280	315	352	391	432	**2017**
175	208	243	280	319	360	**1585**
31	64	99	136	175	216	**721**
						8646

(F) grids flipped, 90 degree rotation to the left

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

1	7	13	19	25	31
2	8	14	20	26	32
3	9	15	21	27	33
4	10	16	22	28	34
5	11	17	23	29	35
6	12	18	24	30	36

1	14	39	76	125	186	**441**
14	**64**	126	200	286	384	**1074**
39	126	**225**	336	459	594	**1779**
76	200	336	**484**	644	816	**2556**
125	286	459	644	**841**	1050	**3405**
186	384	594	816	1050	**1296**	**4326**
441	**1074**	**1779**	**2556**	**3405**	**4326**	**13581**

This time, the 90 degree rotations would be different, as there is a pattern of square numbers running along alternate diagonals, with a repeat either side. I have highlighted this to try and make it clear.

(G) grids flipped, 90 degree rotation to the right

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36

36	30	24	18	12	6
35	29	23	17	11	5
34	28	22	16	10	4
33	27	21	15	9	3
32	26	20	14	8	2
31	25	19	13	7	1

36	60	72	72	60	**36**	**336**
245	232	207	170	**121**	60	**1035**
442	392	330	**256**	170	72	**1662**
627	540	**441**	330	207	72	**2217**
800	**676**	540	392	232	60	**2700**
961	800	627	442	245	36	**3111**
3111	**2700**	**2217**	**1662**	**1035**	**336**	**11061**

So I was correct, grid A is the biggest total, and grid B the smallest. There are many interesting connections between triangle, tetrahedral and square pyramidal numbers.

At the beginning I explained that square pyramidal numbers are linked to triangular numbers. Here is a formula that shows the relationship:

$$P_n = \frac{T_n\,(2n+1)}{3} \qquad\qquad P_{36} = \frac{T_{36}(73)}{3} = \frac{666 \times 73}{3} = 16206$$

Square pyramidal numbers are actually the sum of consecutive pairs of tetrahedral numbers. The *nth* tetrahedral number is the sum of the first *n* triangular numbers (imagine a triangular based pyramid).

Triangular Numbers	**1**	**3**	**6**	**10**	**15**	**21**	**28**
Tetrahedral numbers	**1**	1+3 **=4**	1+3+6 **=10**	1+3+6+10 **=20**	1+3+6+10+15 **=35**	1+3+6+10+15+21 **=56**	1+3+6+10+15+28 **=84**
Square Pyramidal numbers	**1**	1+4 **=5**	4+10 **=14**	10+20 **=30**	20+35 **=55**	35+56 **=91**	56+84 **=140**

This formula shows the relationship between tetrahedral numbers and square pyramidal numbers:

$$Te_n + Te_{(n-1)} = \frac{n(n+1)(n+2)}{6} + \frac{(n-1)n(n+1)}{6}$$

$$Te_{36} + Te_{35} = \frac{36 \times 37 \times 38}{6} + \frac{35 \times 36 \times 37}{6}$$

$$= 8436 + 7770$$

$$= 16206$$

These are lots of interesting relationships to be found and I'm glad I was able to solve this.