Filling the Gaps

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64
65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88
89	90	91	92	93	94	95	96
97	98	99	100	101	102	103	104
105	106	107	108	109	110	111	112
113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128
129	130	131	132	133	134	135	136
137	138	139	140	141	142	143	144
145	146	147	148	149	150	151	152
153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168
169	170	171	172	173	174	175	176
177	178	179	180	181	182	183	184

The dark gray cells represent the odd number squares, the light gray squares represent the even number squares.

Patterns

1. The light gray squares all fall under the category of either 4mod8 or 0mod8.

Proof

Any even number can be written as 2n, where n is an integer.
Therefore any even number squared can be written as $(2n)^2=4n^2$
Therefore any even number squared contains a factor of 4, and is therefore divisible by 4.

2. The dark gray squares all fall under the category of 1mod8.

Proof Part 1

Any odd number can be written as 2n-1, where n is an integer.
Therefore any odd number squared can be written as $(2n-1)^2=4n^2-4n+1$

Therefore any odd number squared contains a factor of 4 with remainder 1.

Proof Part 2

From the table you can see the pattern that the difference between each of the odd squares is divisible by 8 to give a whole number.

<table>
<tr><td>x</td><td colspan="2">1</td><td colspan="2">2</td><td colspan="2">3</td><td colspan="2">4</td><td colspan="2">5</td><td colspan="2">6</td><td colspan="2">7</td><td colspan="2">8</td><td colspan="2">9</td></tr>
<tr><td>x^2</td><td colspan="2">1</td><td colspan="2">4</td><td colspan="2">9</td><td colspan="2">16</td><td colspan="2">25</td><td colspan="2">36</td><td colspan="2">49</td><td colspan="2">64</td><td colspan="2">81</td></tr>
<tr><td>difference.</td><td></td><td colspan="2">3</td><td colspan="2">5</td><td colspan="2">7</td><td colspan="2">9</td><td colspan="2">11</td><td colspan="2">13</td><td colspan="2">15</td><td colspan="2">17</td><td></td></tr>
<tr><td>sum.</td><td></td><td colspan="4">8</td><td colspan="4">16</td><td colspan="4">24</td><td colspan="4">32</td><td></td></tr>
</table>

This is because;
The difference between two squares separated by 2, x^2 and $(x+2)^2$ is $(4x+4)$, which would mean that, for an odd value of x, '2y+1', the value of 4x+4 is 8y+8.
Therefore the difference between any odd square is a factor of 8.

Conclusion

Combining each of these two segments of the proof shows that any odd square number is divisible by eight with a remainder of 1, and will therefore always be in the first column of the 8-wide number grid.

The positioning of the sums of 2 squares

In mod 8, the possible remainders of the squares are 0, 1 and 4, as we saw in the previous proofs.

x^2Mod8	x^2Mod8	sumMod8
0	0	0
0	1	1
0	4	4
1	1	2
1	4	5
4	4	0 (8)

Therefore in the sum of 2 squares we could potentially have a combination of;
As you can see in the table, the only possible columns the sums of 2 squares could be in are column 1, 2, 4, 5, and 0(8).

The positioning of the sums of 3 squares

x^2Mod8	x^2Mod8	x^2Mod8	sumMod8
0	0	0	0
0	0	1	1
0	0	4	4
0	1	1	2
0	1	4	5
0	4	4	0(8)
1	1	1	3
1	1	4	6
4	4	4	4

From that table you can see that when summing 3 squares, you can get an answer in 7 of the columns, however you will never get an answer in the 7^{th} column, as you cannot make 7 out of 0, 1 and 4 using only 3 additions.

QED, probably