Interpolating Polynomials

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Existence

I will want to eventually connect the points

$$(a, A), (b, B), (c, C), (d, D), (e, E), etc.$$

(I know the problem suggests using sub-scripted x's and y's, but I find the final formula much easier to understand if the coordinates are in this format)

As the problem recommends, my initial equation to fit (a, A) to a polynomial will be

$$y = A$$

For a polynomial to connect two points, my formula is this

$$y = \frac{A(x-b)}{(a-b)} + \frac{B(x-a)}{(b-a)}$$

Where did this come from?

If you want a function to pass through the coordinate (a, A), $f(a)$ must equal A. And if you want it to also pass through the coordinate (b, B), $f(b)$ must equal B.

If you take the equation $y = A + (x-a)B$ and plug in the value of $x = a$, then it works well, since $y = A + (a-a)B = A$, which is what we wanted to get. However, if you plug in $x = b$, then you get $y = A + (b-a)B$, which is not what we want. What we need to do is remove the A term, and then the $(b-a)$ coefficient of B. If we can manage that, we will end up with $f(b) = B$.

To remove the A term, we can use the same trick we used to remove the B term. So we get

$$y = A(x-b) + B(x-a)$$

This is successful in getting rid of undesirable terms, since when $x = a$, the B term is multiplied by 0 and is eliminated, and when $x = b$, the A term is multiplied by 0 and is eliminated. However, it does leave us with

$$f(a) = A(a-b)$$

$$f(b) = B(b-a)$$

This is where the denominators come in. They cancel out the unwanted terms, leaving just A or B, just like we want. So

$$y = \frac{A(x-b)}{(a-b)} + \frac{B(x-a)}{(b-a)}$$

OK, so this works for two coordinates. What about three?

It is useful to build upon the polynomial we already have. We now need a function that will work as follows:

$$f(a) = A, f(b) = B, f(c) = C$$

First we introduce C into the polynomial, and then we make sure that it doesn't prevent it working for $x = a$ and $x = b$

$$y = \frac{A(x-b)}{(a-b)} + \frac{B(x-a)}{(b-a)} + (x-a)(x-b)C$$

Now, if we plug in $x = c$

$$f(c) = \frac{A(c-b)}{(a-b)} + \frac{B(c-a)}{(b-a)} + (c-a)(c-b)C$$

There are a lot of things wrong with this at the moment! First, let's remove the first two terms using a now-familiar trick.

$$y = \frac{A(x-b)(x-c)}{(a-b)} + \frac{B(x-a)(x-c)}{(b-a)} + (x-a)(x-b)C$$

And then let's use denominators again to remove the C coefficients when $x = c$.

$$y = \frac{A(x-b)(x-c)}{(a-b)} + \frac{B(x-a)(x-c)}{(b-a)} + \frac{(x-a)(x-b)C}{(c-a)(c-b)}$$

We're not done yet. If we were now to let $x = a$ or $x = b$, this equation no longer works! We need to fix the first two terms of our equation so that these (x-c) coefficients we've just introduced 'disappear' when we input our x coordinates. This will give us our final, 3-coordinate equation.

$$y = \frac{A(x-b)(x-c)}{(a-b)(a-c)} + \frac{B(x-a)(x-c)}{(b-a)(b-c)} + \frac{C(x-a)(x-b)}{(c-a)(c-b)}$$

Ok, so this works for 3 coordinates. What about 4? 5? n?

You should have noticed a pattern forming in the 2- and 3-coordinate equations. If you were to repeat my processes with 4 coordinates, you'd come to an equation with the same patterns again. In general, for n coordinates, each represented as:

$$(X_1, Y_1), (X_2, Y_2), \dots, (X_n, Y_n)$$

An equation passing through each of the coordinates is given by

$$y = \frac{Y_1(x-X_2)(x-X_3)\cdots(x-X_n)}{(X_1-X_2)(X_1-X_3)\cdots(X_1-X_n)} + \frac{Y_2(x-X_1)(x-X_3)\cdots(x-X_n)}{(X_2-X_1)(X_2-X_3)\cdots(X_2-X_n)} + \cdots + \frac{Y_n(x-X_1)(x-X_2)\cdots(x-X_{n-1})}{(X_n-X_1)(X_n-X_2)\cdots(X_n-X_{n-1})}$$

Obviously, the more coordinates you add to your polynomial's path, the more complicated the algebra becomes. This equation can be better summed up in product notation, which I won't use.

Being an A Level student who has done the Numerical Methods 1 unit, I know that this is Lagrange's Interpolating polynomial, and I'm now feeling rather smug because I've just derived it (I think)!

Since this equation can be used for any positive integer value of n, this proves that there always exists a polynomial function that can pass through any list of coordinates, given that their x coordinates are different (Otherwise, it would be a one-one mapping, and therefore not a function).

Using my formula for the four coordinates in the problem,

$$(1,0), (4,6), (2,-2), (0,-2)$$

(and after a bit of fiddly algebra) I got a function

$$f(x) = \left(x - \left(2 + \sqrt{2}\right)\right)\left(x - \left(2 - \sqrt{2}\right)\right)(x - 1)$$

$$f(x) = x^3 - 5x^2 + 6x - 2$$

Uniqueness

Proving "if a quadratic is 0 in 3 places, it is 0 in all places"

The factor theorem states that

$$for\ the\ function\ f(x), if\ f(\alpha) = 0, then\ (x - \alpha) is\ a\ root\ of\ f(x)$$

Therefore, if a quadratic is 0 in three places, i.e.

$$f(\alpha) = 0, f(\beta) = 0\ and\ f(\gamma) = 0$$

Then

$$f(x) = (x - \alpha)(x - \beta)(x - \gamma)$$

This is a contradiction: a quadratic can't have 3 roots. This can be resolved however, because if this function is a quadratic, one of these roots has no value (it equals 0); That way, it only really has 2 roots, and it really is a quadratic. And if one root equals 0, then the whole function would too, no matter the value of x. (NB This is quite a weak argument, I think)

This is a very good way to argue why the theorem is true! I can help explain it a little more clearly though. Note that we might have a scale factor in the function (the leading term has a non-unit coefficient) so with our use of the factor theorem the expression for our supposed quadratic is:

$$f(x) = a_3(x - \alpha)(x - \beta)(x - \gamma)$$

Imagine multiplying this out to get $f(x) = a_3x^3 - a_3(\alpha + \beta + \gamma)x^2 + a_3(\alpha\beta + \beta\gamma + \gamma\alpha)x - a_3\alpha\beta\gamma$. Since $f(x)$ is a quadratic; the coefficient multiplying the x^3 term must be zero. But whatever the values of α, β, γ, the coefficient is a_3. This is a contradiction unless $a_3 = 0$, as you stated: a quadratic can't have three roots. So the function is in fact the constant function $f(x) = 0$.

Alternatively, you can consider the graph of a quadratic function. The maximum number of turning points a quadratic can have is 1. Therefore, it follows that it can only have 3 types of contact with the x axis

- It crosses the axis twices
- It meets the axis, but doesn't cross over
- Or it doesn't cross at all

There is a fourth type which occurs when, in a quadratic written $ax^2 + bx + c$, $a = 0$, but this isn't strictly speaking a quadratic.

This number of turning points arises from the derivative of a quadratic, a linear function, only having one root, as it were. For a function to have two turning points, it would need it's derivative to have two roots, i.e. be a quadratic. This would lead to the function itself being cubic (the integrated partner of a quadratic). And since two turning points are required for the function to be equal to 0 at least 3 times, this means that no non-zero quadratic can have three points all with y coordinate 0.

This excludes the quadratic $y = ax^2 + bx + c$ where $= 0, b = 0, c = 0$, i.e. $y = 0$ for all values of x. (NB I might have rambled a little…)

Prove that "if two quadratic curves share 3 coordinates, then they are the same curve"

Let

$$f(x) = lx^2 + mx + n$$

$$g(x) = rx^2 + sx + t$$

If they cross at 3 places, $\left(a, f(a)\right), (b, f(b)), (c, f(c))$

Then $f(a) = g(a), f(b) = g(b), f(c) = g(c)$

It is therefore obvious that $f(a) - g(a) = 0, f(b) - g(b) = 0, f(c) - g(c) = 0$

If we wrote out the function $(f(x) - g(x))$, we would get the quadratic

$$f(x) - g(x) = (l - r)x^2 + (m - s)x + (n - t)$$

This new function is a quadratic that is 0 for 3 different x coordinates. As we proved earlier, this means that that $f(x) - g(x) = 0$ for all values of x.

This in turn means that $f(x) = g(x)$, meaning that the two functions are identical for all input values. This thus shows that if two quadratic functions share 3 coordinates, then they are indeed the same function.

The logic in my proof could be extended to n order functions, with the statement

"if two nth order curves are equal at (n+1) places, then they are in fact the same curve"

Here we go:

If an nth order polynomial is 0 at $(n+1)$ places, it must be 0 at every x coordinate. This is due to the fact that in order to cross the x axis in $(n+1)$ places, you need a derivative of the nth order, and therefore need a function of the $(n+1)th$ order. This wouldn't meet our requirements, unless we used a zero function ($f(x)=0$), where all y values are 0. Therefore, if an nth order polynomial is 0 in $(n+1)$ places, it is 0 in all places.

If we have two nth order functions, $m(x)$ and $n(x)$, so that:

$$m(x) = p_1x^n + p_2x^{n-1} + \cdots + p_nx + p_{n+1}$$

$$n(x) = q_1x^n + q_2x^{n-1} + \cdots + q_nx + q_{n+1}$$

Where p_i and q_i are constants, and they cross $(n+1)$ times at coordinates

$$\big(\alpha_1, m(\alpha_1)\big), \dots, (\alpha_{n+1}, m(\alpha_{n+1}))$$

It follows that

$$m\big(\alpha_j\big) = n(a_j)$$

For all integers j, $1 \le j \le (n+1)$

And we can create a new nth order function

$$m(x) - n(x) = (p_1 - q_1)x^n + \cdots + (p_n - q_n)x + (p_{n+1} - q_{n+1})$$

Which crosses the x axis $(n+1)$ times. As discussed earlier, if an nth order polynomial is 0 at (n+1) places, it is zero at all places, and it follows that

$$m(x) - n(x) = 0$$

And

$$m(x) = n(x)$$

So these two curves are one and the same, proving the statement.

I hope this has all made sense…

Wow Michael, this is fantastic! Thank you so much for a clear, concise and very neatly presented solution.