WHY PLAY *I SPY* WHEN YOU CAN DO MATHEMATICS?

Robert Andrews and Paul Andrews have some conversations about mathematics.

The following is a story of Robert's mental engagement with mathematics. It has been jointly written by Robert, who is a Y8 student, and his father, Paul, who is a mathematics educator. It starts with Robert, who presents the context for the story. His account, written in his words, is in blue. Paul's commentary draws on what he has written.

I live in Kent with my mother, my sister and my step-dad. My dad lives near Cambridge; so every two or three weeks my mum drives up to Thurrock Services and my dad collects me to take me to his home. Depending on the traffic, these journeys take anywhere between two-and-a-half and four hours. So you can imagine that we talk a lot. We have long conversations about how school's going and how I'm doing, but often we have fun with maths. My dad sets a problem and I try to solve it. It seems a bit weird, I know, but it's fun and fills the time on such long journeys. Recently I've been set much harder problems and this is the latest. We've been talking about it for more than two journeys and this is how it goes.

It's difficult to describe the importance of these journeys. Robert has spent two-thirds of his life travelling between two homes, and long journeys have become a way of life. They allow us to catch up and, without distraction from the *Playstation*, enable us to talk about anything and everything. He is used to travel; even when it means getting up at six-thirty on a Sunday morning to get to Kent for his football team's ten o'clock kick off.

I forget how many times we have played the game where one of us starts with a four-letter word and the other makes another just by changing a single letter – so 'cord' becomes 'card', 'card' becomes 'curd', 'curd' becomes 'curt', and so on. The first person not to be able to make a word from the previous without repetition is the loser. At times, these games have lasted the best part of a hundred miles, but eventually, almost without fail, Robert returns to mathematics and I have to find another problem with which to engage him.

We started off just trying to find prime pairs (two primes that differ by 2); eg, 5 and 7, 11 and 13, 17 and 19. I noticed that, with the exception of 3 and 5, the number in the middle of the pair will always be a multiple of 6. I thought about it for a little while before I knew this had to be true, because the middle number had to be even and because in every three consecutive numbers there is always a multiple of three.

Robert has been playing with primes for a number of years and is aware of the ways in which they are distributed around the multiples of four and six. He is familiar with the ways in which multiples behave and how different sets of multiples, as in games like *Fizz Buzz*, coincide in well-defined and predictable ways. Consequently, his reasoning was not unexpected, although it is always exciting when a learner is able to draw on earlier knowledge and use it to warrant a new conjecture. What he neglected to write concerns his orally articulated awareness that in order for the two primes to be odd, the intervening number had to both even and a multiple of three, since no prime greater than three can be divisible by three. For me, this raises a fascinating issue in terms of children's emergent understanding of the syntax of mathematical reasoning; Robert's oral summary, as presented in the car, was a model of clear thinking, while his written account, as presented above, clearly reflects what he regards as important and leaves out those bits that he perceives as not worthy of recording. What does this mean for our work with children in the classroom? To what extent is the pursuance of syntactical accuracy important? My instinct is that we can always return to the syntax once we have worked on the reasoning; the alternative doesn't seem to make sense – why would I engage with syntax when I have nothing to say?

After this, we started looking at prime triples (three primes differing by 2); for example, 3, 5 and 7. I'm not sure why, but very quickly I felt that this was the only prime triple that differed by 2. To prove this, I had to look at many examples to see if I was right. At the end, I told my dad my reasoning; it was something like this: in order to get three consecutive primes you need three evens. Therefore you have six consecutive numbers. In a set of six numbers you must have 2 multiples of three. One of these will be in the evens and one in the odds, because they differ by three. Therefore there can be no other prime triples differing by 2.

I thought his logic one of the most exciting pieces of mathematical reasoning I had heard from a twelve-year-old in almost thirty years of teaching. His initial instinct, that there was only one possibility, coupled with his confidence that it must be true, was sufficient to push him to think through why. It took him a few minutes, but his solution was entirely unprompted – I was too busy negotiating a contra-flow on the M11 to pay much attention to what he was doing!

Once we had solved that problem, my dad came up with a new one: what prime triples are there if we can have the gap as large as we want and start with any prime? For example, with a gap of 4 we found 3, 7 and 11, but I knew there could be no more for the same reasons as when the gap was two. With a gap of 6 we found 5, 11 and 17, and we found 3, 11 and 19 for a gap of 8. I thought a gap of 10 would be hard because it didn't feel right, but eventually realised I could have 3, 13 and 23. Now I thought I could have any size of gap. We found 5, 17 and 29 for a gap of 12 and 3, 17 and 31 for a gap of 14. Then we hit a problem, because we couldn't find any triples with a gap of 16. The more we looked, the more we felt it was impossible, but I couldn't see why. I knew that adding an even number to any odd number would still give me an odd number, which was what I wanted for a prime. The more I looked, the more it seemed that I would always get a multiple of 3 whenever I tried a gap of 16; 3, 19 and 35 had a multiple of 3; 5, 21 and 37 had a multiple of 3; 7, 23 and 39 had a multiple of three. So it seemed we had a rule, but there was a problem – I still couldn't explain why we always got a multiple of 3. No matter what our starting number, it seemed that you couldn't get two more primes with a gap of 16. Then we realised how stupid we had been; it wasn't just 16, it was multiples of 16. Most of what I have written about happened on one journey, although we continued on a second journey to work out why 16 didn't work.

This was a problem I made up on the spur of the moment, although I suspected that it would lead to further conjectures and proofs. Eventually, with a little prompting, Robert noticed that 16 is one more than a multiple of three and repeated his understanding (although he has elected not to discuss it here) that, with the exception of three itself, all primes are either one less than a multiple of six or one more than a multiple of six. So, he said, if I start with a prime that is one less than a multiple of six and add a gap of 16, which is one more than a multiple of three, then I will get a multiple of three, because the ones cancel. For example, $5 + 16 = 21$, $11 + 16 = 27$, $17 + 16 = 33$ and so on. Similarly, he explained why starting with a prime that was one more than a multiple of six would give a multiple of 3 after two additions of 16; for example $7 + 16 + 16 = 39$.

By now, I thought we had finished, because I believed that because we couldn't get prime triples with a gap of 2, except for the first one, we wouldn't be able to get any triples with gaps bigger than 2 after the first one. Then my dad suggested I looked at a gap of 6 again. This is what he does. He lets me think I've finished and then gives me a new way of looking at the same problem. Sometimes I think this is good and sometimes I just want to talk about something else. But usually he is able to tell which mood I'm in and not push me when I don't want it. Although I like doing mathematics, and get very pleased when I can see why something works, I don't always want to do it.

Although this is something of an aside, it is hardly surprising that Robert is not always in the mood, and would rather do something different from the task I propose. After all, this is in an informal learning situation, and I think it would be detrimental to our relationship for me to insist that he continue. However, the formal classroom is different; I have a responsibility to ensure my students learn. So how do I acknowledge students' affective responses when I teach in the formal rather than the informal situation? Should I allow learners to opt out temporarily to re-engage later when better motivated? More generally, and distancing myself from the question, could teachers allow this without feeling their authority challenged? I suspect there isn't a simple answer, although I also suspect there's an obvious consequence of too many heavy-handed insistences of participation.

I had already found one triple with a gap of 6 and then worked through the list of primes in my head to see if there were any more. I found this difficult, so I got an old envelope out of the glove box and wrote all the primes up to a hundred. Soon I noticed that we could also have 7, 13 and 19; 11, 17 and 23; 17, 23 and 29; 31, 37 and 43 and more. Now I thought that there were probably infinitely many triples with a gap of 6. Also, we could get bigger sets than just 3 primes – 5, 11, 17, 23 and 29 gave a set of 5, and there were several sets of 4 – 41, 47, 53 and 59; 61, 67, 73 and 79. Now my dad asked me if I could explain what was going on and asked me if I could remember anything about the ways that prime numbers were arranged. We talked about this for a little while and then I went back to the fact that primes are always 1 before or 1 after a multiple of 6. So then I wrote the two lists of primes – those that were 1 after a multiple of 6 and those that were 1 before a multiple of 6. Now the picture was clear. Suppose I started with a prime that was 1 less than a multiple of 3. No matter how many 6s I added, the number would never be a multiple of 3, and so I could still get primes in my list. I knew it wouldn't always work, because other numbers could be factors, but I knew that 3 could never be. I also knew now that gaps like 12, 18 and any other multiple of 6 ought to be possible.

I am not trying to present Robert as something he is not. He's an ordinary kid who likes football and prefers to spend most of his weekend mornings in bed, although he does understand that playing football for his school team affords him sufficient status to engage with mathematics without being branded a 'boff'. He is not exceptional, although he would probably make most schools' gifted and talented programme. He tells me he is lucky not only because he has learned more about number theory and geometry than his peers, but also because he enjoys trying to prove the results he obtains, and this really is the rub. If he can do such things, and take pleasure and pride in so doing, then so can others. Unfortunately, the mathematics we do during our journeys together, while of intellectual integrity, does not fit within the skills-driven expectation of the English national curriculum and the long-standing traditions of English classrooms.

Robert understands that mathematics is a unique form of knowledge, based on its deductive reasoning. He understands that without proof mathematics shifts from certainty to speculation. He knows that mathematics makes different knowledge claims from statistics and science. In short, he has an understanding of mathematical epistemology, albeit emergent, which is more secure and explicit than that of many applicants to the PGCE course on which I work. Why? Because English education deals not in knowledge and the nature of knowing but in facts and skills. This explains, for example, why the Right gets vexed when students cannot recite the list of English monarchs in chronological order; it confuses facts with knowledge as if they are the only genuine outcomes of education. Robert finds much of what he is asked to do at school unnecessarily repetitive, particularly when he understands not only how to perform the technique under scrutiny but also why it works and how it relates to other areas of the subject.

Embedded in our story is a series of tasks with which I believe the majority of secondary students could engage meaningfully. The tasks justify why, for example, we teach elementary number theory, because, for most children, the learning of multiples, factors and primes is purposeless and rarely related to anything meaningful, least of all the multiples, factors and primes themselves and their relationships to all integers. Further, the opportunity it gives for children to explore deductive reasoning, proof and the nature of knowing provides an intellectual warrant for the study of mathematics that is lacking in English curricular documents. Robert has already decided he does not want to study mathematics at university and I think it is sad that he has made that decision so early. He knows where he wants to go – Newcastle, because he perceives the city as not only exciting and dynamic but also the spiritual home of his beloved football team. It is a shame that he does not think the same about school mathematics, but he is still young and may change his mind.

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