GENERIC PROOFS: SETTING A GOOD EXAMPLE

Tim Rowland

Why proof?

Any discussion of proof tends to be somewhat aimless unless it includes consideration of the purposes of proof. So what is proof *for*? One view might be that proof is something of a luxury: anyone can apply Pythagoras' theorem to work out how far a leaning ladder would reach up a wall, without knowing why the theorem $c^2 = a^2+b^2$ is true. Or, in 'investigation mode', we might notice that the number of different ways of ascending a flight of 2, 3, 4, 5 stairs taking one or two steps at a time is 2, 3, 5, 8. We might correctly conclude that the number of ways for any flight of stairs can be found by extending the 'pattern' – the Fibonacci sequence. Why should we want to prove either of these things? The first is assured in a thousand books, and more: the first few cases seem to put the 'stairs' generalisation beyond reasonable doubt.

A view of mathematics as a fundamentally rational domain of human activity would regard the scenarios given above as mathematically incomplete. This is not to say that one cannot apply a formula without knowing why it 'works', but that an experience of mathematics that lacks the 'why' dimension misses out the essence of the subject. So one reason for proof is knowing why, and one of the purposes of proof is explanation. There are other purposes – conviction is one of them – but explanation is the one that motivates me now. As a matter of fact, conviction is not the problem. As Reuben Hersh has written:

> *In the classroom, convincing is no problem. Students are all too easily convinced. Two special cases will do it.* [1] (p396)

Thanks to Jill and Robin

All these reflections and speculations might be brought to life by a couple of examples. I have a little bundle of my own, but a double spread in the latest MT conveniently supplies me with some fresh material. In one delightful article, Jill Russell [2] writes about the ways that she and a Y9 class tackled an 'integer triangles' problem: how many triangles are there with given perimeter? Jill confesses that she underestimated the complexity of the problem before offering it to her class, and how she ended up spending many hours working on it. It turns out that the sequence of numbers of possible triangles is 'the same' for even triangles as for odd, although the evens 'lag behind' the odds. Part of the art of Jill's account is that she gives away very little of the detail – the reader has to do some work! Well, I did – though not as much as Jill – and below is how the evens and odds progress.[1]

The sequence of possibilities in the bottom row is strange indeed, and one to add to my repertoire of cautionary patterns – in any investigation, don't be seduced by the first few examples! In the article,

perimeter (odd)	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61	63
perimeter (even)	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	42	44	46	48	50	52	54	56	58	60	62	64	66
number of triangles	**1**	**1**	**2**	**3**	**4**	**5**	**7**	**8**	**10**	**12**	**14**	**16**	**19**	**21**	**24**	**27**	**30**	**33**	**37**	**40**	**44**	**48**	**52**	**56**	**61**	**65**	**70**	**75**	**80**	**85**	**91**

it is Tim (another Tim) who persists with perimeter 15 to refute the first conjecture that there are $\frac{1}{2}(n-3)$ triangles for the odd perimeters from 5 onwards. Jill's comment that

> *3 examples are enough to prove anything in mathematics, especially in the classroom* (p14)

echoes that of Reuben Hersh above. The frame of mind that takes a few examples to be sufficient evidence for all cases is known as 'naïve empiricism'.

She continues, 'I could now spot a pattern, which allowed me to predict further results'. Up to this point, the exploration and the reasoning has been inductive – generating examples, looking for regularities in the data, making and articulating conjectures. But *she doesn't leave it at that*. She talks to colleagues, she interprets the problem in terms of lattice points in a three-dimensional region, and in time 'I saw geometrically *why* there was that connection with the pattern of odds with the evens lagging behind'. Moreover, she comments that 'None of this came in a flash of inspiration'. There's been some talk recently about mathematics as a 'creative' discipline, and of 'creative moments' in mathematical activity. Jill reminds us that such moments rarely, if ever, just happen: they have to be striven for, and they are all the more wonderful for that when – and if – they come. One challenge for Jill, which she does not mention in the article, is how on earth she might make her amazing insights into the problem accessible to her Y9 pupils. What started, for her, as a mathematical problem, is now a decidedly pedagogical one. The challenge and the potential reward is enormous in both cases.

Facing Jill's article is Robin Stewart's review of ***Eight days a week*** [3]. One of the puzzles that Robin pulls out for attention is as follows. 'If the final score in a football match is 3-2, how many different half time scores are there?' Robin lists a few, and finds that there are 12 altogether. Of course, the real problem is not the one stated, but generalising to the case when the final score in x–y (or, for me, m–n: I like to reserve the tail of the alphabet for real variables). Robin comments that students will 'hopefully discover that if you multiply x and y and then add $x+y+1$, you will get the correct answer. Now $xy+x+y+1$ factorises to $(x+1)(y+1)$. A really nice investigation from a very simple question'.

I do agree, and it reminds me of one generalisation that comes out of the 'stamps problem' – what is the greatest amount that cannot be made using a supply of m pence and n pence stamps (m and n coprime)? These generalisations – inductive inferences – are harder to come by than, say, that for triangular numbers because they involve functions of two variables rather than one. The skilled problem solver knows to hold one variable fixed whilst investigating the other. Another kind of skill is needed to recognise that the function, for the football scores and for the stamps, must be symmetrical in the two variables.

But what is missing from Robin's brief account – which is not to say that he didn't consider including it – is the question 'Why?'. In this case, and I speak only for myself, the generalisation itself gives compelling guidance towards a possible explanation as to why it might hold for all x and y, and the 'Aha!' insight comes with considerably less expense than it did for Jill's perimeter problem. If the final score was x–y, the first team could have any score from 0 to x at half time, giving $x+1$ possibilities in all. Similarly, there are $y+1$ half time possibilities for the second team, irrespective of the first team's score at half time. The conclusion now follows from a particular understanding of multiplication of positive integers in terms of Cartesian product: if Teddy has 4 tee-shirts and 3 pairs of shorts, how many different outfits can he wear?

What is so strange and unsettling, once I 'see' this, is that the problem appears to be so trivial that I would hesitate to offer it to students in the first place. Well, it would depend, wouldn't it? We're now looking at the problem from the end of a road that we've travelled, beginning with induction and ending with deduction; but many students will have to travel the same road before they arrive at the same place. Experience suggests that the generalisation is far from obvious, the obviousness being with the benefit of hindsight.

So the same pedagogical question arises as before. How might we make the explanation available to students who do not generate it for themselves, or how might we direct them towards constructing it?

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Generic examples

Here, I return to a theme that I touched on in MT167 [4], the claim that certain truths of a general nature can be conveyed in arguments embedded in particular 'generic' examples, particular instances of the general statement.

As an illustration, take the general statement that arises as a conjecture from the half time scores investigation: 'If the final score is m–n, there are $(m+1)(n+1)$ possible half time scores'. Of course, the problem is very accessible to many pupils, especially in KS2, who might express the generalisation without using those literal conventions, but in a prose sentence such as 'You add one to the scores of both teams and multiply[2] them together'. It's important to emphasise that a generic example of a general statement is not merely a confirming instance of it, but a chain of reasoning about the particular example. I don't think I can improve on Nicolas Balacheff's succinct statement of the way this works:

> *The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class.* [5, p.219]

So, I *could* take the final score 3–2, write down all the possible half time scores, count them and find that there are 12, and observe that $(3+1)\times(2+1)$ also equals 12. This is a confirming instance of the generalisation (otherwise it would be a counterexample). It makes the conjecture a little more credible, but so far no argument has been built around it. It's not (yet) a generic example, because it gives no insight into why the $(m+1)(n+1)$ statement might be true for *all* positive values of m and n. One way of constructing a generic argument based on this particular case might be to present the following partial table of possibilities, to be completed by pupils:

		Half time goals: 1st team (A)			
		0	1		
Half-time goals 2nd team (B)	0				
	1	0-1			

An alternative, arguably preferable, approach, involving the pupil or the class in the construction of the argument, might be to pursue a line of questioning such as:

How many goals could the first team (A) have scored at half time?

Suppose team A hadn't scored any goals at half time. How many goals could team B have scored? How many possibilities is that? What if team A had scored one goal at half time?

Now this might well lead to a sense of why the number of possibilities for $3-2$ is $3+3+3+3$. But perhaps you suspect, as I do, that the presence of the 3 in A's final score is a possible distraction. The '3' in $3+3+3+3$ is there, not because it happens to be team A's score at the final whistle, but because it's one more than B's score. So although it happened to be the example given in the investigation 'starter', it has that built-in drawback that makes it less than ideal as a generic example. Without recapitulating the generic argument here, I'd be more inclined to discuss the case for 4–2 or even 5–2. The point is that *the choice of generic example is not arbitrary*: some examples work better than others do; they carry and convey the generalisation rather better because the salient operations on the variable(s) can easily be tracked through the argument.

Now the idea of a generic example is that we can somehow 'see' beyond its particularity to what might happen in other instances. The transparent presentation of the example is intended to enable transfer of the argument to other instances. Ultimately the audience can conceive of no possible instance in which the analogy could not be achieved. But this is a *psychological* phenomenon. How can we know that it has taken place in the mind of the pupil? One way is to ask pupils to talk about what would happen in another particular case (or cases). For example, what would be the possibilities at half time if the final score were 9–3?

There remains a vexed question to do with the transition from generic example to conventional symbolic proof, to the argument framed in x and y (or m and n) without reference to particular instances[3]. Having internalised the argument in a particular generic example, and having perceived its structural relationship to other possible examples, how can students be enabled to write 'proper' proofs, to bridge the gap between the generic understanding and the general exposition? The transition between the first kind of knowing and the second seems to entail the harnessing of ideas to notation. I am bound to reiterate what I have said and written elsewhere, that if explanation has already been achieved through a generic argument of the kind above, then the purpose of symbolic proof must be something other than knowing why. It follows, in such circumstances, that symbolic proof is redundant if its purpose is explanation. Some colleagues react very badly to this suggestion, and I have some sympathy with those who argue, with a wider range of objectives in mind, that that a significant part of becoming a mathematician lies in developing the ability to write and understand formal proof, and not just the ideas behind the proof. Incidentally, it would be interesting to know whether Jill Russell came to 'see' why the pattern of odds and evens is the way it is in her perimeter problem through reasoning with particular cases. My guess would be that it was, and that she felt that this yielded the insight she was striving for. Did she then feel the need to write out the general case with perimeter n and sides a, b and c? In fact I have done so in similar circumstances, but have seen it as an exercise in notation, and not for the purpose of strengthening conviction.

Gauss and the sum 1+2+3+… +100

The story is told about the child C. F. Gauss, who astounded his village schoolmaster by his rapid calculation of the sum of the integers from 1 to 100. Whilst the other pupils performed laborious column addition, Gauss added 1 to 100, 2 to 99, 3 to 98, and so on, and finally computed fifty 101s with ease. The power of the story is that it offers the listener a means to add, say, the integers from 1 to 200. Gauss's method demonstrates, by generic example, that the sum of the first $2k$ positive integers is $k(2k+1)$. Nobody who could follow Gauss' method in the case $k=50$ could possibly doubt the general case. It is important to emphasise that it is not simply the fact that the proposition $1+2+3+\ldots+2k = k(2k+1)$ has been verified as true in the case $k=50$. It is the *manner* in which it is verified, the form of presentation of the confirmation.

Paul Hoffman (drawing on E. T. Bell's ***Men of mathematics***) recounts the story in his recent best-seller ***The man who loved only numbers*** [6]. His comment on it (quoting mathematician Ronald Graham) is a telling testimony to the genericity of Gauss' method.

> *What makes Gauss' method so special . . . Is that it doesn't just work for this specific problem but can be generalised to find the sum of the first 50 integers or the first 1,000 integers . . . or whatever you want.* (p.208)

In introducing the notion 'generic example' to audiences of all kinds, I have frequently chosen Gauss' method as a paradigm of the genre: generic among generic examples!

Postscript

In MT167, I mentioned the video ***Teachers count*** which, ostensibly, promoted the three-part lesson structure subsequently adopted by the national numeracy strategy. In the middle phase of one lesson, some Y5/6 children investigate the well-known 'jailer problem'[4] in small groups, before being brought together by Kate, their teacher. The solution turns out to hinge on the fact that every square number has an odd number of factors. In fact, Kate explains this to the class by reference to a generic example, although no reference is made in the commentary to this aspect of her teaching and proof strategy. She points out that every factor of 36 has a distinct co-factor, with the exception of 6, and so it must follow that 36 has an odd number of factors.

Her choice of 36 is interesting – small enough to be accessible with mental arithmetic but with sufficient factors to be non-trivial. It happens to be the least square with more than one prime factor. How conscious was Kate of this in choosing to work with 36 rather than, say, 25? In any case, it demonstrates again that the choice of a generic example is not arbitrary, but a conscious pedagogical act. In fact 100 would be just as good. 144 has rather too many factors (counting factors has much in common with the half-time scores problem!). 196 or 225 would be good choices were it not for the fact that Year 5/6 children might find it difficult to locate their factors. So everything considered, 36 takes some beating.

It would be nice to have more examples of this kind, and I would be delighted to hear from any readers who recognise the approach to proof through careful reasoning with particular examples. I would particularly be interested to know about their own examples of the method, and their experiences of using it in the classroom with pupils of any age.

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Notes

1 Once upon a time, when the world was young, there was a computer programming language called BASIC that you could use to work out things like this.

2 More likely, 'times', but that's another issue, and not the focus of what I want to write about here.

3 Of course, not all general mathematical arguments need to be expressed algebraically in the sense of using literal variables, although many students seem to have been led to believe that they ought to.

4 A certain prison has 100 prisoners, 100 cells and 100 jailers. One prisoner is assigned to each cell. One night, when the prisoners are all locked away, the first jailer unlocks all the cells. Then the second jailer locks all the cells whose numbers are multiples of 2. Next, the third jailer changes the state of all the cells that are multiples of 3, and so on through to the 100th jailer. The jailers then all fall asleep. Which prisoners were able to escape from their cells?

References

1 Reuben Hersh 'Proving is convincing and explaining'. ***Educational studies in mathematics*** 24, 1993, pp. 389-399.

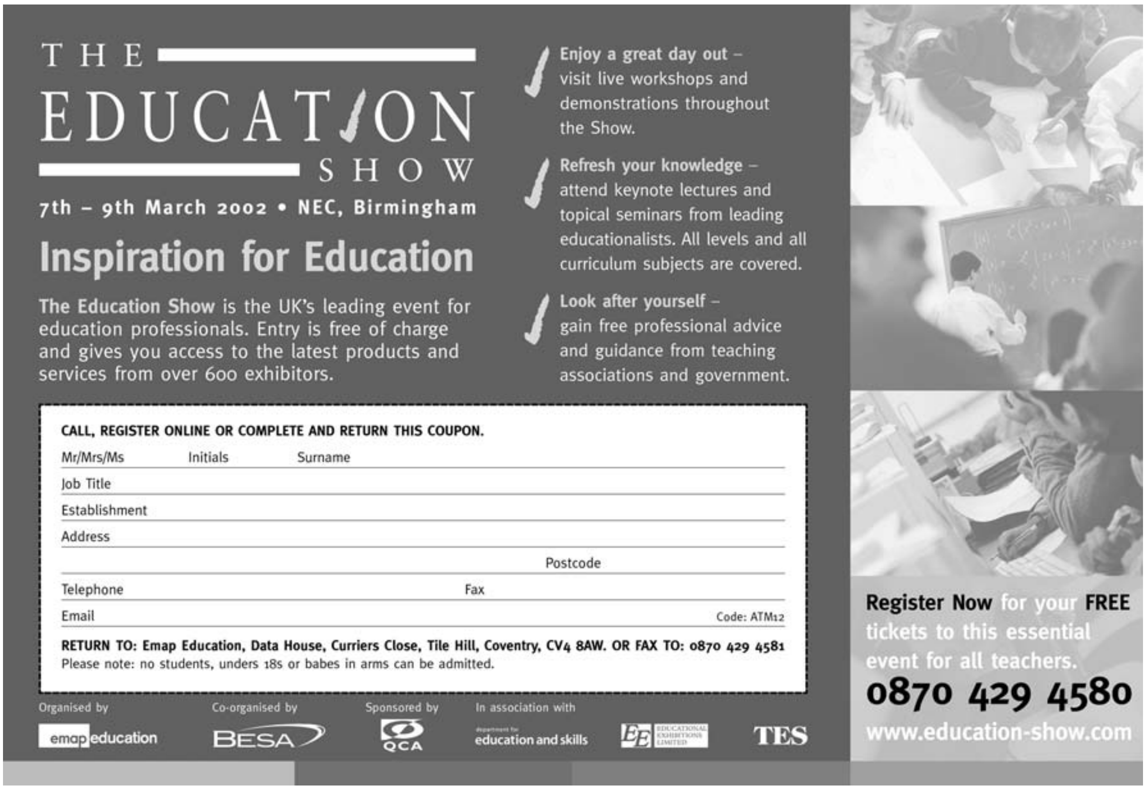
2 Jill Russell: 'A lesson for pattern spotters'. MT175, 2001, pp.14-15

3 Robin Stewart: 'Lessons for eight days a week'. MT175, 2001, p.15

4 Tim Rowland: 'i is for induction'. MT167, 1999, pp.23-27.

5 Nicolas Balacheff: 'Aspects of proof in pupils' practice of school mathematics' in Pimm, D.(ed) ***Mathematics, teachers and children***, London, Hodder and Stoughton, 1988, pp.216-235.

6 Paul Hoffman: ***The man who loved only numbers***, Fourth Estate.



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