'i' IS FOR INDUCTION

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I've recently been thinking about different kinds of problem-solving activities, how students have responded to them and what they have learned from engaging with them. I'll begin by asking you to take five minutes to consider, if you will, *how you might go about* tackling each of the following problems. (Feel free to put the rest of my article aside, if any of them *really* interests you).

1 ***Boundary***: An integer-sided square is subdivided into a grid of unit squares. How many unit squares lie on the boundary?

2 ***Diagonal:*** How many unit squares lie on the diagonals of the above square grid?

3 ***Fifteen:*** Choose a positive whole number and write down all of its factors, including the chosen number and 1. Now add all the digits of those factors (that sum would be 5 if you had chosen 13). Repeat the whole process on the new number. Keep going. Try different starting numbers.

4 ***Stairs:*** In how many different ways can you ascend a flight of stairs in ones and twos?

5 ***Partitions:*** The number 3 can be 'partitioned' into an ordered sum of (one or more) positive numbers in the following four ways: 3, 2+1, 1+2, 1+1+1. In how many ways can other positive numbers be partitioned?

6 ***Sums of squares:*** In how many different ways can a prime number be written as a sum of two square numbers? What about non-primes?

7 ***Polygram:*** A polygram is constructed by joining alternate vertices of a polygon with straight lines (the best-known example is a pentagram). At each vertex of the polygram, an internal angle is formed between adjacent edges. What is the sum of these angles of a polygram?

8 ***Painted cube:*** A cube is made up from lots of little cubes, and the surface is painted red. How many little cubes have three painted faces? Two faces? One? None?

Your response to each problem, your heuristic for solving it, will depend on a number of factors. One of these will be whether you have encountered and worked on the same problem before. If you have, it may offer little challenge to you and probably little interest. Another factor, I suggest, is whether you feel you have a secure analytical overview of the situation presented in the problem. By that, I mean whether you feel able to approach it with some general case in mind.

That's how I described my initial reaction to *Painted cube* in MT143 [1]. I feel the same about *Boundary*. I can 'see' a *general* square. I 'count' the non-corner boundary unit squares ('side minus 2', 4 times) and add the 4 corner squares. Another person might prefer to collect some numerical data on the boundaries of particular square. Example – a 4-by-4 square has 12 unit squares on the boundary. Try it with some children in your class. What do they choose to do? Was that what you expected? More difficult – how will you constrain their choices (or try to avoid constraining them) by the way you present the problem, the materials you offer them, the recording methods you suggest (or do not), or the prior judgements you make about how they are likely to approach the problem?

I have included *Polygram* for my own benefit. I invented the problem (without claim to originality) but have not yet worked on it, and so I can think aloud as I write. I have no instant sense of how it might develop. I know that for a regular pentagram, the angles add to 180°, but that's all. Well, not quite. I *suspect* that the angles of every pentagram will have the same sum. I know that a hexagram consists of two distinct triangles, and so the angle sum will be 360°. And so, almost despite myself, I have one conjecture about polygrams in general. But, unlike with *Boundary*, I had to examine some data, some particular cases, to gain a sense of what might be the case for *any* polygram.

For me, *Diagonal* (which I also devised as I started to write) lies somewhere between *Boundary* and *Polygram*. I anticipate that the 'rules' for even- and odd-sided squares will be slightly different, because only the latter have a 'middle' unit square on both diagonals. Ah . . . it wasn't as obvious as *Boundary*, but I have a growing sense of the general case.

I doubt whether anyone who has not worked on (or read about) *Sums of squares* would have a feel for what it's about without generating some data, such as $53=2^2+7^2$ (unique apart from order). The number 59 cannot be so expressed, whereas $65=1^2+8^2$ or 4^2+7^2.

Induction

Some, but not all, of the problems listed above seem to invite the search for *regularity among examples*, in order to make predictions about other particular cases (such as the sum of the angles of a 10-gram) and to arrive at conjectures of a more general kind. The word 'conjecture' captures the idea that knowledge of the general case, or any case beyond the data in hand, is for the moment provisional, tentative. The process of arriving at such a conjecture from a finite data-set is *induction* (or inductive inference), with a small 'i'. Unfortunately, mathematicians are conditioned to associate 'induction' with Mathematical Induction (with a capital 'I'), which is a schema for a particular kind of proof about propositions to do with all natural numbers.

I recall, at school, summing cubes of integers to observe that each partial sum was the square of a triangular number as a prelude to Proof by Mathematical Induction.

$$1^3+2^3=3^2,\ 1^3+2^3+3^3=6^2,\ 1^3+2^3+3^3+4^3=10^2\ldots$$

The proof-process (Induction) was carefully named, but not the process (induction) of arriving at the statement to be proved. I am sure little has changed. Given the current concern for 'proper' mathematical vocabulary [2], perhaps the time has come to introduce 'induction' into the language (and practice!) of the mathematics classroom.

I am speaking of induction here as a scientist would, in relation to discovery or invention. Inductive reasoning takes the thinker beyond the evidence, by somehow discovering (by generalisation) some additional knowledge inside themselves. The mechanism which enables an individual to arrive at plausible, if uncertain, belief about a whole population, an infinite set, from actual knowledge of a few items from the set, is mysterious. The nineteenth-century scientist William Whewell captures the wonder of it all:

> Induction moves upward, and deduction downwards, on the same stair [. . .]. Deduction descends steadily and methodically, step by step: Induction mounts by a leap which is out of the reach of method. She bounds to the top of the stairs at once [. . .]. [3, p114]

Here deduction is portrayed in terms of descent, just as a syllogism is presented on the written page – methodical, steady, safe, descending. By contrast, induction is framed as daring, creative, ascending. Whewell discusses the symbiotic relationship between induction and deduction. They must be 'processes of the same mind'. Without induction there is nothing to justify by deduction; but it is the business of deduction, writes Whewell, to 'establish the solidity of her companion's footing'.[1]

Why should we want to draw attention to induction by naming it in the classroom? Because to celebrate induction is to highlight the humanity of mathematics and the character of mathematical invention. I call two witnesses in support of this claim.

> Analysis and natural philosophy owe their most important discoveries to this fruitful means, which is called induction. Newton was indebted to it for his theorems of the binomial and the principle of universal gravity [4, p176].

> The purpose of rigour is to legitimate the conquests of the intuition. (Hadamard, quoted by Burn [5, p1]).

On a personal note, I would add that some of the work that has given me the greatest pleasure to write (and thanks frequently to MT) to publish, has consisted of accounts of the inductive muse at work in myself.[2] The consequence has been a powerful desire to offer opportunities to the students I have taught in school and at university, to experience the same delight and sense of mathematical 'one-ness'.

Proof and naïve empiricism

Lest I be misunderstood, it is important to remark that there is more to mathematics than induction. Believing is not the same as knowing. Indeed, induction carries the danger of premature conviction.

Recently, I worked on *Partitions* with some primary PGCE students. The students accumulated data about partitions of 2, 3 and 4, and observed a doubling pattern. Some checked that it extended to partitions of 5. They were finished! Those who formulated this as $P(n)=2^{n-1}$ were more than finished, and well satisfied. "But how do you *know*", I asked, "that it will continue to double *every* time?" The typical response was along the line, "Well, it has up to now, so it seems reasonable to suppose that it always will". This is naïve empiricism at work. As Reuben Hersh has remarked,

> In the classroom, convincing is no problem. Students are all too easily convinced. Two special cases will do it [7, p396].

The teacher's invitation to scepticism about patterns seems like rather an empty gesture. Most of the time, a few special cases *do* point the way to eternity.[3] The point of the teacher's proof-provoking question is not to achieve certainty – which is already assured – but to encourage the quest for insight. As Hersh says, the primary purpose of proof in the teaching context is to *explain*, to illuminate *why* something is the case rather than to be assured that it is the case.

There is a genuine problem here, especially but by no means exclusively with younger pupils. So often, explanation is of a quite different order of difficulty from inductive conviction. Number theory, in particular, is so amenable to conjecture yet so resistant to proof. For example (see *Sums of squares*), Fermat's proof by descent that primes of the form $4k+1$ can be uniquely expressed as the sum of two squares presents a challenge even to mathematics undergraduates.

At least the theorem (about expressing primes as sums of squares) is of major significance. This is more than can be said of the (admittedly curious) outcome of *Fifteen* – the conjecture that every such sequence arrives and remains fixed at 15 sooner or later. In a recent issue of *Equals*, *Fifteen* is commended as a good Key Stage 2 whole-class starter [8]. I would agree that it offers a motivating context for work on divisors and gives rise to a nice tree of sequences. Beyond that, I feel very uncomfortable with it, because there seems to be little prospect of either pupil or teacher being able to explain *why* each sequence should include 15[4]. There is a plethora of such 'chain' investigations, such as *Happy and sad numbers*, which are to do with summing the squares of the digits of an integer, and iterating on that sum.[5] For me, *Fifteen* is uncomfortably reminiscent of the notorious Thwaites' Conjecture,[6] a dead-end for the classroom if ever there was one.

I am appealing for teachers, in choosing problem starters, to have a sense of whether inductive conjecture is almost certain to be a terminus for the investigation. Is that what they want? Every time? What message will that give about the nature of mathematics, apart (hopefully) from being good fun? What price the 'Aha!' of insight, of rational, connected mathematical knowledge?

Ideally, we might hope for pupils to generate their own explanations. More often, we might attempt to point them towards ways of perceiving the problem situations that have the potential to prompt explanatory insight. I believe that the 'generic example' can play a crucial role in this latter respect. The generic example is a confirming instance of a proposition, carefully presented so as to provide insight as to *why* the proposition holds true for that single instance.

> The generic example involves making explicit the reasons for the truth of an assertion by means of operations or transformations on an object that is not there in its own right, but as a characteristic representative of the class [10, p219].

I have argued elsewhere [11] for the unrealised pedagogic potential of generic examples, and will settle for one topical illustration here.

The 1997 Ofsted video *Teachers count* features one teacher, Kate, with a class of 10- and 11-year-olds. In the middle phase of the 'Numeracy hour' lesson, the children investigate the *Jailer problem* in small groups, before being brought together by Kate (presumably for some direct teaching). The solution turns out to hinge on the fact that every square number has an odd number of factors. In fact, Kate explains this to the class by reference to (what we recognise as) a generic example. She points out that every factor of 36 has a distinct co-factor, with the exception of 6, and so it must follow that 36 has an odd number of factors. She then generalises, 'One of the factors of a square number is a number times itself (*sic*); that's why it's a square number, isn't it?' Her choice of 36 is interesting – small enough to be accessible with mental arithmetic but with sufficient factors to be non-trivial. Not surprisingly, no reference is made in the commentary to this aspect of her teaching and proof strategy.

Empirical and structural generalisation

Inductive inference has to start with examples, but if investigation finishes with 'spotting a pattern' (or even stating a formula) it remains at the level of naive empiricism. As my students said, 'Well, it has up to now, so it seems reasonable to suppose that it always will'. Liz Bills and I have tried to distinguish between two kinds of generalisation.[7]

- Empirical – that which 'merely' generalises from tabulated numerical data
- Structural – deriving from an overview of the situation from which the data arises.

> ...We emphasise that one form of generalisation is achieved by considering the form of results, whilst the other is made by looking at the underlying meanings, structures or procedures [12].

Empirical generalisation may in time become structural; if knowing *that* becomes knowing *why*, when an explanation for what is observed becomes available. Empirical generalisation is authentic, important but incomplete, mathematics. My opening problem set was chosen to exemplify some situations which seem to require data to be collected (and typically tabulated) before any progress towards empirical generalisation is possible, as well as others where structural generalisation is an immediate prospect. The response (empirical or structural) is in part a function of the student and the classroom environment determined by the teacher, not solely of the problem in isolation.

Again, I suggest that it is important for teachers to have a sense of how different individuals will respond to a given starting point – empirically or structurally. At different times, both are desirable. If we want students to experience and reflect upon induction, it *might* be better not to offer them *Boundary* – which, in turn, might be ideal for structural generalisation, and for discussion of different but equivalent formulations of such a generalisation.[8] Last year, we began a workshop on mathematical processes with Primary PGCE students by inviting them to work on *Painted cube*. Within minutes, many (by no means all) of them had written things like $6(n-2)^2$ and were asking 'What shall I do now?' They were not 'being difficult'; they were experiencing and demonstrating structural generalisation. This year, we began with *Stairs*. As a stimulus for experiencing and reflecting on induction and (eventually) explanation, it was much more effective.

Train spotting and 'the formula'

I began this article with a set of 'investigation starters'. How subversive I felt in the 1970s, working with such material in the classroom! Time has revealed the reason for my sense of unease: Cockcroft has legitimised our activity, and said that everybody should be doing it. Some of the teachers who enthusiastically promoted the place of investigational work in the curriculum have come to regret the institutionalisation of investigation, not least within GCSE coursework. The unexpected regularity, the creativity and *frisson* of mathematical discovery, has been forced into an algorithmic mould: data-pattern-generalisation (and formula for extra marks). Assessment has hardened the paradigm. In her recent book, Candia Morgan has addressed

> ...some of the tensions and contradictions implicit in the official and practical discourse [of 'investigation']. In particular, the ideals of openness and creativity, once operationalised through the provision of examples, advice and assessment schemes, become predictable and even develop into prescribed ways of posing questions or 'extending' problems and rigid algorithms for 'doing investigations' [13, p73].

Dave Hewitt [14] has famously complained that in children's driven determination to tabulate numerical data, 'their attention is with the numbers and is thus taken away from the original situation'. I personally experience a sense of irritation when students insist on an algebraic 'formula' – in the form of a function $f(x)$ – for everything. The sense of closure once they have it is palpable. Presumably, they are the victims of GCSE coursework indoctrination. My irritation stems, in part, from awareness that a few students will be scared off by the formula, and sense failure because they didn't find it, or feel anxiety because their undoubted rationality is frozen by the sight of it. One reason I like to work on *Stairs* with students is that the sequence (Fibonacci) is easily generated recursively, but the formula for the *n*th term refuses to yield to difference tables and is virtually inaccessible (though students still want to know it!)

Conclusion

There are tensions here, didactic dilemmas. The national trend is towards pedagogic uniformity, a lesson structure and teaching styles that 'work' for all teachers and almost all learners. It is far from clear, on the evidence to date, whether or how this will embrace inductive approaches to learning. In any case, even the one-time proponents of investigation seem to be increasingly disillusioned. Are we, therefore, to consign investigational work to the dustbin of late twentieth century school-maths history, along with multibase arithmetic? That would be neither my conclusion nor my wish, notwithstanding the distorting consequences of GCSE assessment over the last decade. Indeed, I would fan the flickering flame of Paragraph 243 [16], adding four additional bullet points:

Investigational work *at all levels* should include opportunities for:

- induction
- explanation
- empirical generalisation
- structural generalisation

So long as we hold onto belief in pupils' entitlement to experience and learn *mathematics*, I would want to emphasise and promote the centrality of generalisation in the school mathematics curriculum, and 'investigation' as a valuable and authentic means whereby students might encounter and experience it.

Notes

1 Whewell personifies Induction and Deduction (like the characters of Bunyan's ***Pilgrim's progress***) as though they were two characters inhabiting the mind of the scientist. It is gratifying to note, moreover, that Whewell does not conform to the stereotype and make Induction female (illogical, intuitive, uncertain, apt to lead, to seduce her companion, capable of error) and Deduction male (logical, secure, the steadying influence on his partner). In Whewell's text, both characters are portrayed as female.

2 If I had to choose just one of my *own* articles to take to a desert island, it would be [6].

3 In the case of *Partitions*, naïve empiricism and confidence in doubling can be challenged by working on a different problem. Points are marked unevenly on the boundary of a circle, and each point is joined to every other point. How many regions are formed?

4 It makes much more sense simply to investigate sums of divisors – try 2, 5 and then 10.

5 By 'happy' coincidence, an article on the topic [9] appeared in the ***Mathematical Gazette*** in the same

week that I submitted the first draft of this article to MT. Alan Beardon's ingenious resolution of conjectures about cycles and fixed points confirmed what I had suspected: that this is fascinating but non-elementary mathematics.

6 If N is even, halve it; otherwise multiply by 3 and add 1. Brian Thwaites' prize of £1000 to anyone who can prove (or refute) the conjecture – that every such sequence is ultimately attracted to 1 – remains unclaimed.

7 Liz first introduced the distinction and the terms in her thesis [18].

8 See, for example, Bob Vertes' handling of Picture Frames (aka *Boundary*) with a Year 7 class, on the video which accompanied Open University course EM235, *Developing mathematical thinking*.

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