Solution to Particularly General

For any real number x:

$$[(1-x)(1+x+x^2+x^3) = 1+x+x^2+x^3-x-x^2-x^3-x^4$$
$$\Leftrightarrow (1-x)(1+x+x^2+x^3) = 1-x^4$$

For any real number x:

$$(1-x)((1+x)(1+x^2)(1+x^4)) = (1-x^2)(1+x^2)(1+x^4)$$
$$\Leftrightarrow (1-x)((1+x)(1+x^2)(1+x^4) = (1-x^4)(1+x^4) = (1-x^8)$$

For the last expression, we can first remember that for a a real number,

$$sin(2a) = 2cos(a)sin(a)$$

That's why for $sin(2a) \neq 0$, $cos(a) = \frac{sin(2a)}{2sin(a)}$
Now we can evaluate the expression:

$$cos(\frac{x}{2})cos(\frac{x}{4})cos(\frac{x}{8})cos(\frac{x}{16}) = \frac{sin(x)sin(\frac{x}{2})sin(\frac{x}{4})sin(\frac{x}{8})}{16sin(\frac{x}{2})sin(\frac{x}{4})sin(\frac{x}{8})sin(\frac{x}{16})} = \frac{sin(x)}{16sin(\frac{x}{16})}$$

Which is true only when $sin(\frac{x}{16}) \neq 0 \Leftrightarrow$ x is not a multiple of 16π

Now that we have prove those three particular expressions, let's evaluate the general forms.
We can make the assumption that: $(1-x)(1+x+x^2+x^3+...+x^n) = 1-x^{n+1}$
Let's prove it. By developping, we notice:

$$(1-x)(1+x+x^2+x^3+...+x^n) = 1+x+x^2+x^3+...+x^n-x-x^2-x^3-x^{n+1} = 1-x^{n+1}$$

And we get the answer.

Regarding $(1-x)((1+x)(1+x^2)(1+x^4)...(1+x^{2^n}))$, we can make the assumption that it is equal to $1-x^{2^{n+1}}$.
Let's prove this by induction. We have the statement P(n): $(1-x)((1+x)(1+x^2)(1+x^4)...(1+x^{2^n}))$
For n = 0, $(1-x)(1+x) = (1-x^2)$, then P(1) is true.
Let's suppose P(n) is true.
We have: $(1-x)((1+x)(1+x^2)(1+x^4)...(1+x^{2^n})) = (1-x^{2^{n+1}})$
$(1-x)((1+x)(1+x^2)(1+x^4)...(1+x^{2^n})(1+x^{2^{n+1}})) = (1+x^{2^{n+1}})(1-x^{2^{n+1}}) = (1-(x^{2^{n+1}})^2) = (1-x^{2^{n+2}})$
Then we get that if P(n) true, then P(n+1) is true, thus by the axiom of induction, we conclude:
$(1-x)((1+x)(1+x^2)(1+x^4)...(1+x^{2^n})) = 1-x^{2^{n+1}}$

Finally, regarding $cos(\frac{x}{2})cos(\frac{x}{4})cos(\frac{x}{8})cos(\frac{x}{16})...cos(\frac{x}{2^n})$, we can conjecture that it equals to $\frac{sin(x)}{2^n sin(\frac{x}{2^n})}$
We have the statement P(n): $cos(\frac{x}{2})cos(\frac{x}{4})cos(\frac{x}{8})cos(\frac{x}{16})...cos(\frac{x}{2^n}) = \frac{sin(x)}{2^n sin(\frac{x}{2^n})}$
For n=1, $cos(\frac{x}{2}) = \frac{sin(x)}{2sin(\frac{x}{2})}$
Let's suppose P(n) is true, we have: $P(n+1) = P(n)cos(\frac{x}{2(n+1)})$
$\Leftrightarrow P(n+1) = \frac{sin(x)}{2^n sin(\frac{x}{2^n})}cos(\frac{x}{2(n+1)})$
$\Leftrightarrow P(n+1) = \frac{sin(x)}{2^n sin(\frac{x}{2^n})}\frac{sin(\frac{x}{2^n}}{2sin(\frac{x}{2(n+1)})}$
$\Leftrightarrow P(n+1) = \frac{sin(x)}{2(n+1)sin(\frac{x}{2(n+1)})}$
P(n+1) is true, thus by the axiom of induction $cos(\frac{x}{2})cos(\frac{x}{4})cos(\frac{x}{8})cos(\frac{x}{16})...cos(\frac{x}{2^n}) = \frac{sin(x)}{2^n sin(\frac{x}{2^n})}$