Summing Geometric Progressions

Summing the first 15 terms of 3, 9, 27, 81, 243,…

This sequence is equivalent to $3, \quad 3\times 3, \quad 3\times 3^2, 3\times 3^3, \ldots$

For this sequence, $S = 3 + 3\times 3 + 3\times 3^2 + \cdots + 3\times 3^{14}$

And $3S = 3\times 3 + 3\times 3^2 + 3\times 3^3 + \cdots + 3\times 3^{15}$

So $3S - S = 2S = 3\times 3^{15} - 3$

$$S = \frac{3\times 3^{15} - 3}{2} = 21523359$$

Summing the first 12 terms of 5, 10, 20, 40, 80,…

This sequence is equivalent to $5, \quad 5\times 2, \quad 5\times 2^2, 5\times 2^3, \ldots$

For this sequence, $S = 5 + 5\times 2 + 5\times 2^2 + \cdots + 5\times 2^{11}$

And $2S = 5\times 2 + 5\times 2^2 + 5\times 2^3 + \cdots + 5\times 2^{12}$

So $2S - S = S = 5\times 2^{12} - 5 = 20475$

Summing the first 20 terms of 3, 3x2, 3x2x2,…

The above statement is equivalent to the $\sum_{i=1}^{20} 3\times 2^{i-1}$

For this sequence, $S = 3 + 3\times 2 + 3\times 2^2 + \cdots + 3\times 2^{19}$

And $2S = 3\times 2 + 3\times 2^2 + 3\times 2^3 + \cdots + 3\times 2^{20}$

So $2S - S = S = 3\times 2^{20} - 3 = 3145725$

Summing the first 10 terms of $^1/_2, {}^1/_4, {}^1/_8, {}^1/_{16}, \ldots$

This sequence is equivalent to $\frac{1}{2}, \quad \frac{1}{2}\times\left(\frac{1}{2}\right), \quad \frac{1}{2}\times\left(\frac{1}{2}\right)^2, \frac{1}{2}\times\left(\frac{1}{2}\right)^3, \ldots$

For this sequence, $S = \frac{1}{2} + \frac{1}{2}\times\left(\frac{1}{2}\right) + \frac{1}{2}\times\left(\frac{1}{2}\right)^2 + \cdots + \frac{1}{2}\times\left(\frac{1}{2}\right)^9$

And $2S = 1 + \frac{1}{2} + \frac{1}{2}\times\left(\frac{1}{2}\right) + \cdots + \frac{1}{2}\times\left(\frac{1}{2}\right)^8$

(I suspect that doing it this way will make later manipulation easier.)

So $2S - S = S = 1 - \frac{1}{2} \times \left(\frac{1}{2}\right)^9 = \frac{1023}{1024} \approx 1$

Summing the nth term of $a, ar, \quad ar^2, ar^3 \ldots$

First, $S = a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$

And $rS = ar + ar^2 + ar^3 + ar^4 \ldots + ar^n$

$rS - S = ar^n - a$

$S(r - 1) = a(r^n - 1)$

$S = \frac{a(r^n - 1)}{(r - 1)}$

This is a generalised expression for finding the sum of any geometric progression, and if I use it for the 4 sums I have already calculated, I should get the same answer

$S_1 = \frac{3(3^{15} - 1)}{(3 - 1)} = 21523359$

$S_2 = \frac{5(2^{12} - 1)}{(2 - 1)} = 20475$

$S_3 = \frac{3(2^{20} - 1)}{(2 - 1)} = 3145725$

$S_4 = \frac{1/2\left(1/2^{10} - 1\right)}{\left(1/2 - 1\right)} = \frac{1023}{1024}$

So this generalisation seems to work.