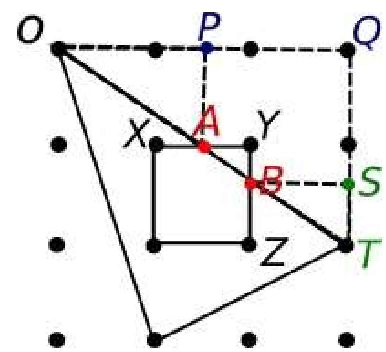
## Stage 4 ★★★
## Mixed Selection 1 - Solutions

**1. Griddy region**

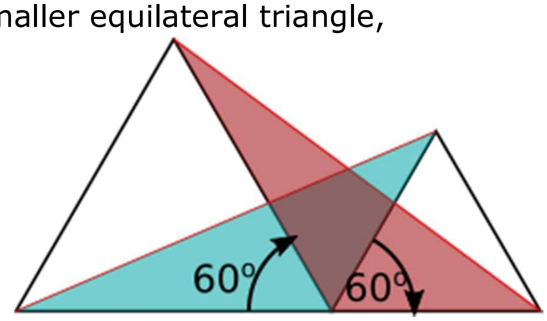
We want to find out how far the points $A$ and $B$ (the points on both the triangle and square) are from $Y$. The triangle $OQT$ is 3 units across by 2 units down. The triangle $OPA$ is similar to $OQT$, and is half its size (since it is one unit down rather than two). So the point $A$ is 3/2 units from $O$, so 3/2 - 1 = 1/2 units from $X$, so 1 – 1/2 = 1/2 unit from $Y$.




Similarly, the triangle $BST$ is 1/3 the size of $OQT$, so the point $B$ is 2/3 units from $Z$, so 1 - 2/3 = 1/3 unit from $Y$. So the triangle $AYB$ has area 1/2 x 1/2 x 1/3 = 1/12 square units, and the area of overlap is 1 – 1/2 = 11/12 square units.

**2. Two equilateral triangles**

Consider the coloured triangles in the diagram below. They both have one side equal in length to the larger equilateral triangle, and these sides are at an angle of 60° to each other. They also both have one side equal in length to the smaller equilateral triangle, and these sides are also at an angle of 60° to each other.




So a 60° rotation will map the blue triangle onto the red triangle. So the two triangles must be congruent. So the third sides, which are the red lengths, must be equal.

*A fuller solution is available at:* https://nrich.maths.org/12950/solution

### 3. Cut-up square

First, we label the corners of the square, the midpoint, and the point at which the two lines intersect $A$, $B$, $C$, $D$, $E$ and $F$ respectively. We also label the four triangles created within the square as 1, 2, 3 and 4.

Call the area of triangle a.

Angle $AFD$ is equal to angle $EFC$ as they are vertically opposite, and angle $FAD$ is equal to angle $FCD$ as they are alternate. So, triangles 1 and 3 are similar. As $E$ marks the midpoint of a side of the square, length $CE$ is half the value of length $AD$.

This means that the area of triangle 3 is equal to 4a, as its base and height are double that of triangle 1.

It also means that length $AF$ is equal to twice the value of length $FC$. If we let $AF$ be the base of triangle 3 and $FC$ be the base of triangle 2, then these triangles have the same height (equal to the perpendicular distance from $D$ to the line $AF$) so the area of triangle 2 is equal to half that of triangle 3 - that is, 2a.

As line $AC$ is a diagonal, the combined areas of triangles 2 and 3 is equal to half the area of the square. 4a + 2a = 6a, so the square has a total area of 12a. The combined area of triangle 1 and triangle 4 is then also 6a. As we have defined the area of triangle 1 as a, we then know that triangle 4 has an area of 5a.

As the ratio of $P$: $Q$ is simply the ratio area of triangle 2:area of triangle 4, we then know that this ratio is 2a:5a=2:5.

**4. Garden fence**

In the diagram below, triangle $ABC$ represents the garden, $CD$ represents the fence and $E$ is the foot of the perpendicular from $D$ to $AC$.

The two sections of the garden have the same perimeter so $AD$ is $10\text{m}$ longer than $DB$. Hence, $AD = 30\text{m}$ and $DB = 20\text{m}$.

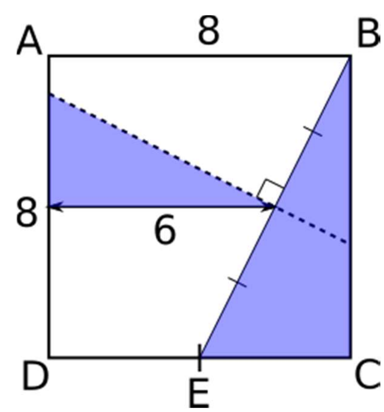
Triangles $AED$ and $ACB$ are similar, so $\frac{AE}{AC} = \frac{AD}{AB} = \frac{30}{50}$. Hence, $AE = \frac{3}{5} \times 30\text{m} = 18\text{m}$.

Also, $\frac{ED}{CB} = \frac{AD}{AB} = \frac{30}{50}$. Hence, $ED = \frac{3}{5} \times 40\text{m} = 12\text{m}$.

Finally, by Pythagoras: $CD^2 = EC^2 + ED^2 = (12^2 + 24^2)\text{m}^2 = 5 \times 12^2\text{m}^2$. So the length of the fence is $12\sqrt{5}\text{m}$.

**5. Folded square**

In the diagram below, the line EB has been added. The dotted line must be the perpendicular bisector of EB, since for B to fold onto E, B and E must be the same distance on either side of the fold.




The two triangles coloured blue are similar, as they both contain a right angle and their other two angles are 90° rotations of each other.

The dotted line and the folded line cross halfway between E and B (since EB is bisected), so 4 cm from the top (and from the bottom) and 2 cm from the right (and from E). So the distance marked on the diagram below is 6 cm.

The sides of the larger triangle are in the ratio 1:2 (from 4:8), so the vertical side of the smaller triangle must be 3 cm. So the dotted line meets AD 4−3 = 1 cm below A.

*A fuller solution is available at:* https://nrich.maths.org/12983/solution

*These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk)*