Stage 4 ★★★
Mixed Selection 1 - Solutions

1. In or out

Let the radii be r. Enclose the circle in a square with sides $2r$.

The unshaded area of the square consists of 4 quadrants (quarters of circles) of radius r.

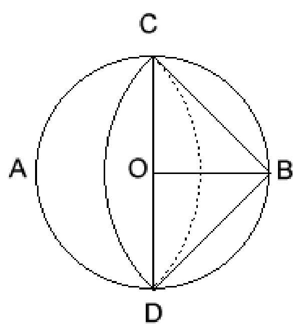
The total area of the square is $4r^2$ and the area of each quadrant is $\frac{\pi r^2}{4}$.

So the shaded area is $4r^2 - \pi r^2 = (4-\pi)r^2$. Therefore, the fraction of the circle that is shaded is $\frac{(4-\pi)r^2}{\pi r^2} = \frac{4}{\pi} - 1$.

2. Snake eyes

Let O be the centre of the circle and let the points where the arcs meet be C and D respectively. $ABCD$ is a square since its sides are all equal to the radius of the arc CD and $\angle ACB = 90°$ (angle in a semicircle).

In triangle OCB, $CB^2 = OC^2 + OB^2$; hence $CB = \sqrt{2}$ cm. The area of the segment bounded by arc CD and diameter CD is equal to the area of the triangle BCD, i.e.
$\left(\frac{1}{4}\pi\left(\sqrt{2}\right)^2 - \frac{1}{2}\times\sqrt{2}\times\sqrt{2}\right) = \left(\frac{\pi}{2} - 1\right) cm^2$.



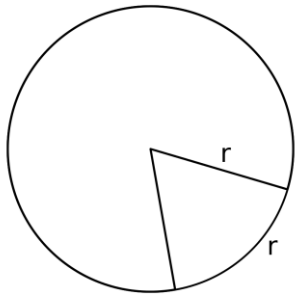
The unshaded area in the original figure is therefore $(\pi - 2)\ cm^2$. Now the area of the circle is $\pi\ cm^2$, and hence the shaded area is $2\ cm^2$.

A fuller solution is available at: https://nrich.maths.org/12997/solution

These problems are adapted from UKMT Mathematical Challenge problems (ukmt.org.uk)

3. Clown hats

Let the radius of the circular piece of cardboard be r. The diagram shows a sector of the circle which would make one hat, with the minor arc shown becoming the circumference of the base of the hat.



The circumference of the circle is $2\pi r$. Since $6r < 2\pi r < 7r$. we can cut out 6 hats in this fashion.

Moreover, the area of cardboard unused in cutting out *any* 6 hats is less than the area of a single hat. Hence there is no possibility that more than 6 hats could be cut out.

4. Roll on

In returning to its original position, the centre of the disc moves around the circumference of a circle of diameter $3d$, i.e. a distance $3\pi d$. As the disc does not slip, its centre moves a distance πd when the disc makes one complete turn about its centre, so 3 complete turns are made.

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