Odds and Evens

Imagine we play the game with a even numbered balls and b odd numbered balls.

Conjecture: a and b are consecutive triangle numbers.

Proof:

	a even	***b odd***
a even	a^2-a	ab
b odd	ab	b^2-b

For a fair game,

$$a^2 - a + b^2 - b = 2ab$$

$$\Rightarrow \quad a^2 - 2ab + b^2 = a + b$$

$$\Rightarrow \quad (a - b)^2 = a + b$$

This proves that:

the total number of balls has to be a square number

the difference between the number of odd and even balls is the square root of the total number of balls.

Let the total number of balls be n^2.

Then $a + b = n^2$

and $a - b = n$

Adding the two equations:

$$2a = n^2 + n$$

$$\Rightarrow \quad a = \tfrac{1}{2} (n^2 + n)$$

$$\Rightarrow \quad a = \tfrac{1}{2} n (n+1)$$

This is the formula for the n^{th} triangular number, so a must be a triangular number.

Similarly, b is the $(n-1)^{th}$ triangular number.