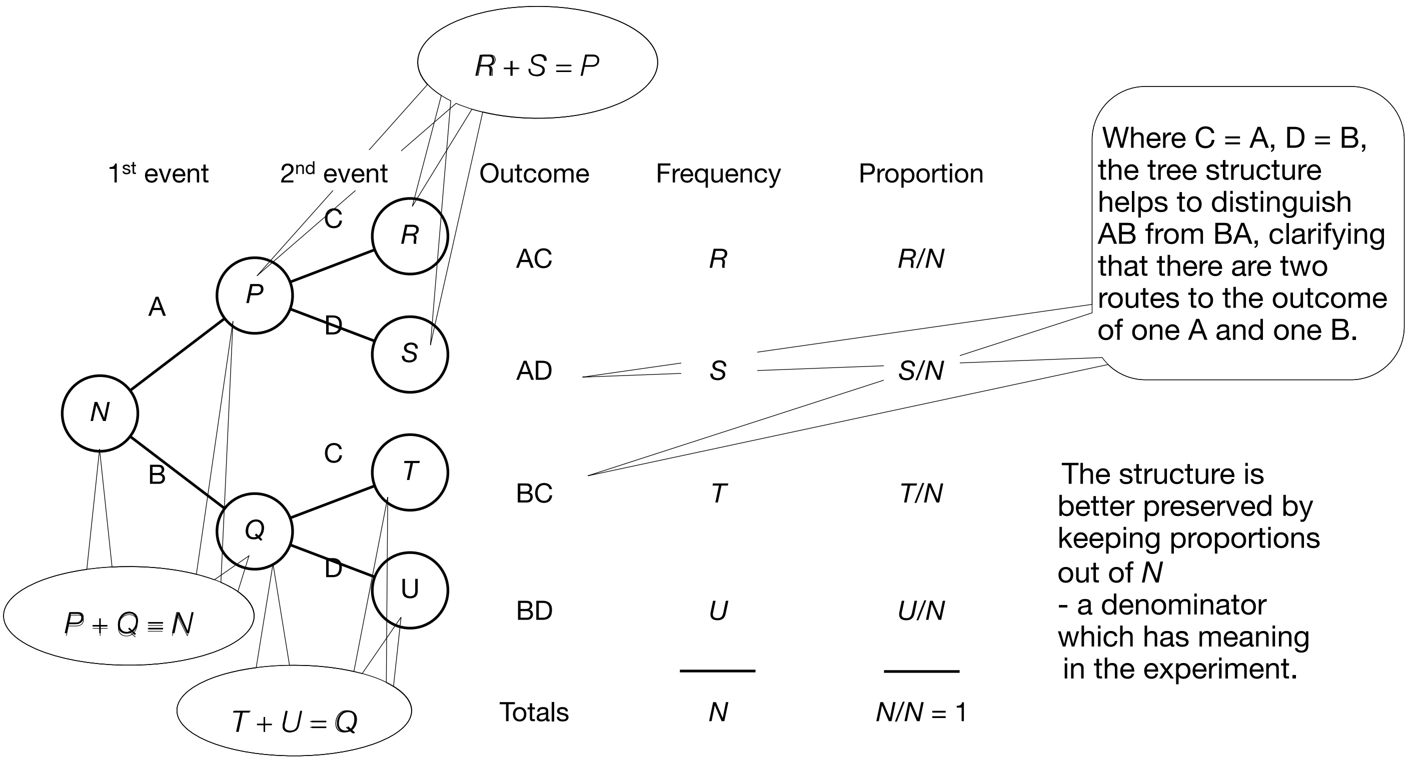
Representations: general analysis

Frequency and probability trees

A tree diagram represents a complete set of mutually exclusive outcomes. All our resources depend on students understanding that a tree (whether a frequency tree or a probability tree) provides alternative narratives: so for instance, first A then C, or first B then C, and so on.

- The progression through different representations is:
  - observed (empirical) frequency tree: a description of a past series of experiments, based on empirical observation
  - expected frequency tree: an expectation of a future series of experiments, based on theoretical reasoning
  - probability tree: a representation of a single future experiment, based on theoretical reasoning
- A frequency tree and a probability tree are structurally the same, in that the first set of branch segments gives the alternatives for the first event, and the subsequent pairs of branch segments give the alternatives for the second event.
- The first event (with alternatives labelled A and B on the tree below) and the second event may have the same chance of occurring, as in *Which Team Will Win?* where the first event is which team scores the first goal and the second is which team scores the second goal. Equally, they could be different (so on the tree below the second event has alternatives labelled C and D), as in *The Dog Ate My Homework*, where the first event is whether the student is being truthful or not, and the second is whether Mr LI Detector accuses them of lying or not.

Frequency tree: expected results



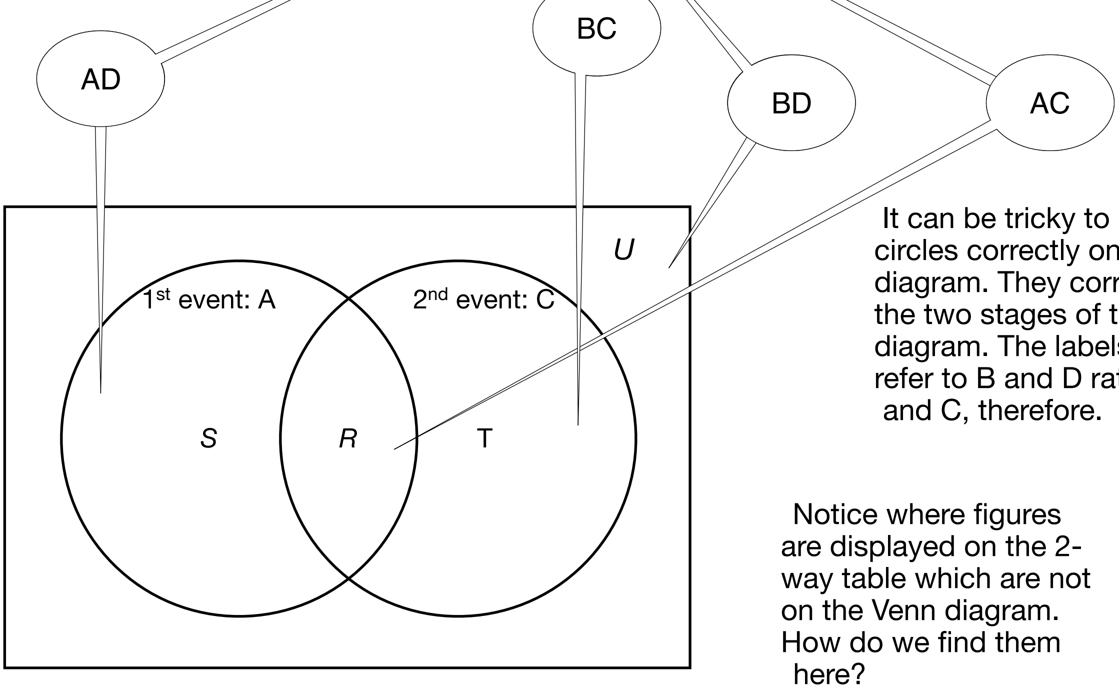
P is the number of the N trials which we expect to have outcome A.
Q is the number of the N trials which we expect to have outcome B.
Similarly, R and T are the numbers from P and Q trials respectively which we expect to have outcome C, and S and U are the numbers from P and Q trials respectively which we expect to have outcome D.

2-way Tables and Venn Diagrams

Expected results

		2nd event		
		C	D	Totals
1st event	A	R	S	$R + S = P$
	B	T	U	$T + U = Q$
	Totals	$R + T$	$S + U$	N

Notice where figures from the tree are displayed on the 2-way table. Which are not on the tree? How do we find them there?



It can be tricky to label the circles correctly on a Venn diagram. They correspond to the two stages of the tree diagram. The labels could refer to B and D rather than A and C, therefore.

Notice where figures are displayed on the 2-way table which are not on the Venn diagram. How do we find them here?

From expected proportions to probability calculations

The expected proportion for a particular outcome of the N trials is equivalent to the probability that it occurs. Working from the expected proportion enables students to see how the probability calculation (multiplying along the branches) mirrors the process by which they obtained the expected frequency of each outcome.

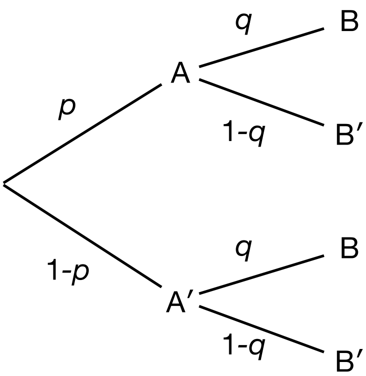
This argument is more transparent when numbers are used rather than letters, so we suggest you refer to the analysis of the individual problems to see examples of this process, eg. *Who Is Cheating?*.

Probability Tree: independent events

Which Team Will Win?, The Dog Ate My Homework, Who Is Cheating? and sampling with replacement in *Prize Giving* are all examples of independent events.

The condition for independence is that P(A∩B) = P(A and B) = P(A) x P(B) = P(B and A) = P(B∩A), where A and A′ (the complement of A, or not A) are the options for the first event, and B and B′ for the second.

1st event 2nd event



Outcome	Expected proportion	Probability
AB	*R/N*	*R/N* = *p* x *q*
AB′	*S/N*	*S/N* = *p* x (1 - *q*)
A′B	*T/N*	*T/N* = (1 - *p*) x *q*
A′B′	*U/N*	*U/N* = (1 - *p*) x *(1 - q)*
Totals	1	1

From the expected frequency tree.

On a frequency tree, the total for each pair of branch segments is equal to the previous value, from which they fork. On a probability tree the sum of the probabilities on each pair of branch segments is 1, and the total probability, summed across all possible sets of branches, is also 1.

In addition, for independent events, the 2-way table entries are determined by the marginal totals:

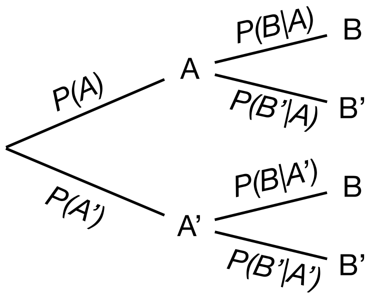
		2nd event		
		C	D	Totals
1st event	A	*R* = *PX/N*	*S* = *PY/N*	*R* + *S* = *P*
	B	*T* = *QX/N*	*U* = *QY/N*	*T* + *U* = *Q*
	Totals	*R* + *T* = X	*S* + *U* = Y	*N*

Probability Tree: dependent events

Sampling without replacement (see *Prize Giving* for a context) provides an example where the expected results for the second event depend upon the outcome of the first event. The tree has the same structure and general results as the tree above, but this time the value of *q* depends on the value of *p*, and it is not the case that *P(A∩B) = P(A) x P(B).*

It is easier to see the difference between the probability trees for independent and dependent events when numbers are used, so we would suggest you look at the analysis of *Prize Giving* to find out more about this.

Conditional Probability



$$P(A \cap B) = P(A) \times P(B \mid A)$$

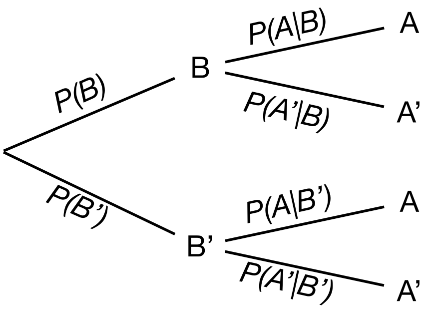
$$P(A \cap B') = P(A) \times P(B' \mid A)$$

$$P(A' \cap B) = P(A') \times P(B \mid A')$$

$$P(A' \cap B') = P(A') \times P(B' \mid A')$$

'Forward' tree - A precedes B

‘Reverse’ tree - B precedes A



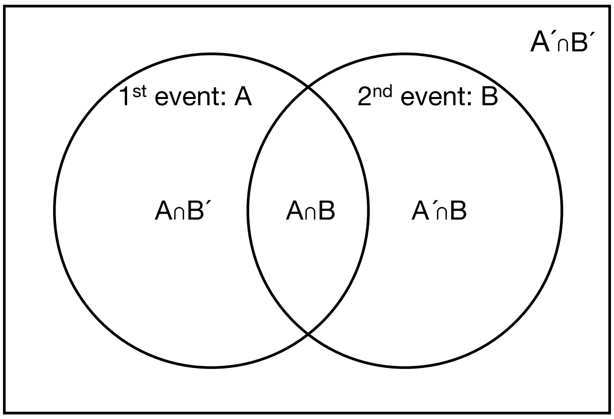
$$P(B \cap A) = P(B) \times P(A \mid B)$$
$$P(B \cap A') = P(B) \times P(A' \mid B)$$
$$P(B' \cap A) = P(B') \times P(A \mid B')$$
$$P(B' \cap A') = P(B') \times P(A' \mid B')$$

Deriving Bayes' Theorem is straightforward once the relationships on the tree diagram are clear:

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$
$$P(A \mid B) = \frac{P(B \cap A)}{P(B)}$$
$$P(A)P(B \mid A) = P(B)P(A \mid B)$$

since P(A∩B) and P(B∩A) are the same.

		2nd event		
		B	B´	
1st event	A	A∩B	A∩B´	A
	A´	A´∩B	A´∩B´	A´
		B	B’	



The ‘reverse’ tree diagram can be drawn up from the ‘forward’ tree diagram or from the 2-way table . In either case, the process is much more transparent if proportions of whole numbers (using the expected results) are used rather than probabilities - see *Who Is Cheating?* for an example. For tree diagrams, the required result using probabilities is:

$$P(A \mid B) = \frac{P(B \cap A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

With this approach, students can answer conditional probability questions with understanding, rather than merely applying a formula.