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TOWARDS A NEW PROBABILITY CURRICULUM FOR SECONDARY SCHOOLS

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All is not well for probability in the school mathematics curriculum. Students typically have little problem with elementary aspects, but many find it very difficult to know how to structure more complex problems. They rarely have well-developed informal methods of grasping the essence of a problem nor do they have any intuition as to whether they are applying the right formula in the right way. Our approach is based on mathematical modelling, in which all stages of the modelling cycle, not just the calculation stage, can be included. We believe giving students problems which model real life helps them to develop a greater range of methods and intuitions than the standard approach does. We also believe that considering multiple representations simultaneously and using whole numbers (natural frequencies) as far as possible help students to grasp the essence of the analysis of a problem. The teacher's role in asking key questions is also a vital component.
Key words: probability, mathematical modelling, multiple representations, natural frequencies

INTRODUCTION

Probability is part of most secondary school mathematics curricula[1] and it is also a component of many science curricula. The language of probability is also part of everyday discourse, used increasingly to create eye-catching headlines and sound-bites. For all sorts of reasons, therefore, our students need to engage with probability at school.

However the problems associated with the teaching and learning of probability are well-documented (Moore, 1997; Pratt, 2011). Over 20 years ago, Garfield & Ahlgren (1988, p. 47) noted a number of reasons for this, including difficulty with proportional reasoning and interpreting verbal statements of problems, conflicts between the analysis of probability in the mathematics lesson and experience in real life, and premature exposure to highly abstract, formalised presentations in mathematics lessons. Teacher knowledge may also be an issue, since not all teachers will have studied probability during their own school education[2] (Papaieronymou, 2009).

This is not just an issue for schools. It is clearly important that people understand what they are told when they are provided with information about medical tests and procedures, for instance. However, research has shown that few people, including doctors, understand the

[1] It is part of the International Baccalaureate, for instance, and the fundamentals of probability are included by TIMSS in its reporting of 8th grade mathematics.
[2] I am aware that this is an issue for teachers in South Africa, for instance, from personal experience of teaching on courses for practising teachers there.

real level of risk when it is presented to them using the language of mathematical probability. It is much easier to understand when the information is given using whole numbers (natural frequencies) (Gigerenzer & Hoffrage, 1995).

We are therefore working on a new approach to the teaching of probability which builds on such research (see also Hoffrage, Gigerenzer, Krauss, & Martignon, 2002; Martignon, Laskey, & Kurz-Milcke, 2007; Schlottmann & Wilkening, 2011).

OUR APPROACH

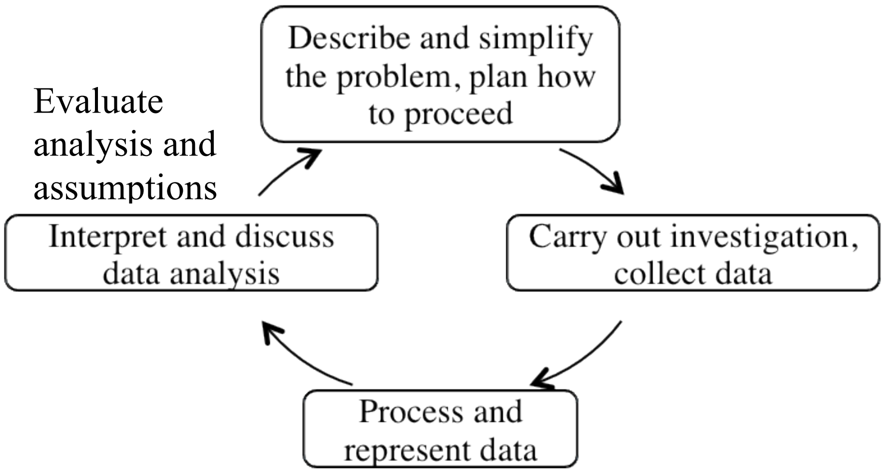


Figure 1. Modelling Cycle

Rather than starting with theoretical concepts of probability, we use mathematical modelling as our basis (Figure 1). This encourages students and teachers to think about solving problems rather than simply calculating answers. It also permits longer, more complex problems to be considered thus reducing the disconnect which many students experience when they move from simple one-step problems to the two-stage problems which typically feature in exam syllabi. It is our experience so far that students enjoy the challenge and, provided lessons are appropriately structured, cope perfectly well.

Figure 2 represents the structure on which our problems are based. The horizontal axis shows the progression from a single trial (n = 1), to a multi-step practical investigation (typically n = 36, if a die is used in modelling the problem), to a computer simulation (n = many thousands or tens of thousands, even). In this framework, the theoretical probability is then presented as the limiting result of a large number of trials. The vertical axis shows how results are represented: a tally chart or table, a tree diagram and a 2-way contingency table.

Introducing the tree diagram right at the start of our proposed curriculum is a particular feature of our approach. The branches can be interpreted as a complete set of mutually exclusive narratives, each leading to one outcome. The focus of a 2-way table is the complete set of mutually exclusive outcomes, but without the sense of an unfolding story. Hence the two forms of representation complement each other, and help students to begin to appreciate how important it is to identify the appropriate sets for the numerator and denominator when expressing a particular outcome as a probability.

The single trial provides one outcome. Comparing their outcome with that of others in the class enables students to appreciate the full range of outcomes which can occur in a given experiment. The multi-step practical will generally consist of 36 trials because this is enough to allow students to see what the distribution of outcomes looks like. Because we often use a die to model random variation in a trial, it also reduces any confusion or lack of understanding caused by non-integer numbers when practical results are compared with theoretically expected results.

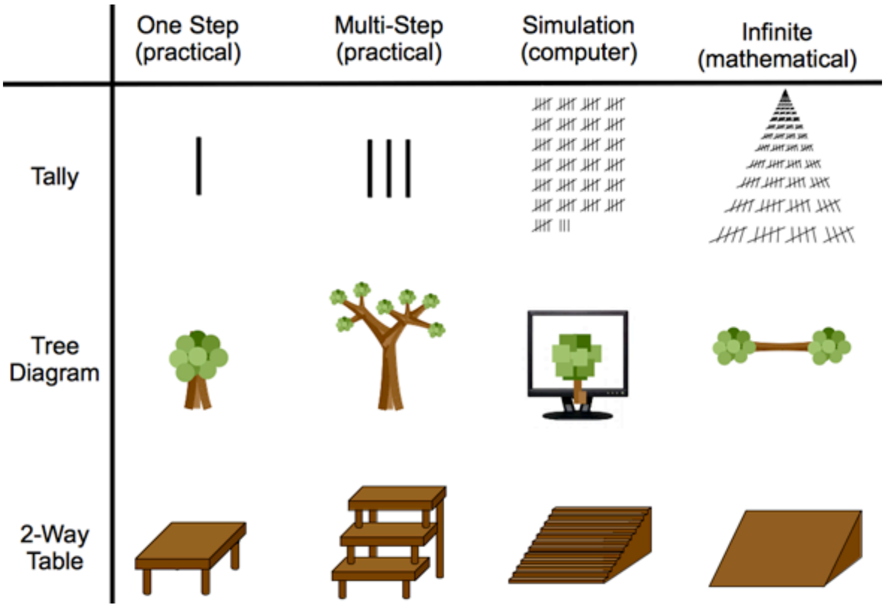


Figure 2. Framework for teaching probability

We intend that, ideally, this stage should be followed by a computer simulation which enables students to investigate how the pattern of results settles down in the long run, although at the time of writing this is still work in progress and so did not form part of the classroom trials described. However, in the trials, class results over 36 trials were aggregated in several lessons, providing some hundreds of results. The results of the practical investigation and computer simulation can be compared with the results that would be expected if n could be increased without limit – and this is, of course, the expected or theoretical result.

As will be seen in the following description of classroom trials of two problems, students between 10 and 14 years of age easily completed the experimental investigations, and were enabled to interpret their results at a surprisingly high level using tree diagrams and 2-way contingency tables, despite these being new to most of them.

SCHOOL TRIALS

Initially, two problems were prepared for use in schools. ‘Which team will win?’ features a football game scenario, and is intended for students just starting to quantify probability, so 11-12 year-olds (Year 7) in England. The second problem, ‘The dog ate my homework!’, is based on a scenario of a teacher who believes he can detect when students are lying about their homework. This is intended for students with more experience of quantifying probability, who have yet to be introduced to conditional probability, so 12-14 year-olds (Year 8 or 9) in England.

Which team will win?
Every weekend, Team Raccoon and Team Beaver play each other at 2-Goal Football – they play until two goals have been scored. Sometimes games only last five minutes, sometimes they seem to go on for ever.

The dog ate my homework!
A certain teacher, Mr L I Detector, claims he can tell when students are lying about their homework. This is true. Unfortunately, he also accuses some students who are actually telling the truth. What are the chances someone will be wrongly accused?

The selection of schools and classes for trial lessons was opportunistic, involving teachers with whom I was already working. The detail of the lesson observations is given in Table 1. There were two trial lessons at each of schools B and C, and one at each of A and D. I discussed the lessons afterwards with students and (separately) teachers.

Table 1. Lesson observations

Problem trialled	School	Description of school	Number / age of students	Number / length of lesson(s)	Level[3] of class
Which team will win?	A	Non-selective, in an urban area with selective grammar schools	30, 11-12	One extended lesson, about 100 minutes	Level 4/5
Which team will win?	B	Non-selective, in a village close to Cambridge	25, 11-12	Lesson 2 of 2, 60 minutes	Mixed ability – level 3 to level 7
Which team will win?	B	As above, different teacher[4]	Not known, 12-13	One 60 minute lesson	Mixed ability – level 3 to level 8

[3] National Curriculum levels for England and Wales: level 4 should be reached by most students on transfer from primary to secondary education, level 8 is achieved by the most able 14 year-olds. http://www.education.gov.uk/performancetables/ks3_04/k3.shtml, accessed 28 March 2012.
[4] I did not observe this lesson, but was sent an evaluation by the teacher by email.

Problem trialled	School	Description of school	Number / age of students	Number / length of lesson(s)	Level[3] of class
The dog ate my home-work!	C	Non-selective, in a village close to Cambridge	31, 13-14	One 50 minute lesson	Level 6/7, described as reasonably high ability, but not a top set
Which team will win?	C	As above, same teacher	27, 10-11	One 60 minute after school session	Small groups of students from local primary schools at an after-school enrichment club
The dog ate my home-work!	D	Non-selective in a village fairly close to Cambridge	32, 13-14	One 60 minute lesson	Level 6/7, top set

The football game was played with 'probability dice', plain dice with red stickers on four faces and blue on the other two, except at School B, where students used dice numbered from 1 to 6. At the start of the lesson, groups were asked to play one 2-Goal Football game, by throwing their die twice and recording the result. Results around the class were then compared. This provided an opportunity to clarify that there are four possible outcomes – RR, RB, BR, and BB – corresponding to a win for the Raccoons, a draw in which the Raccoons score first, a draw in which the Beavers score first, and a win for the Beavers.

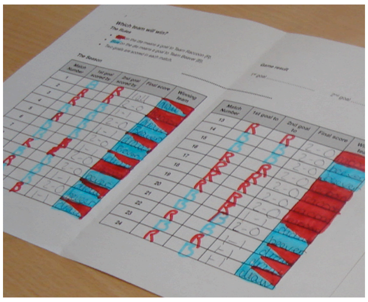
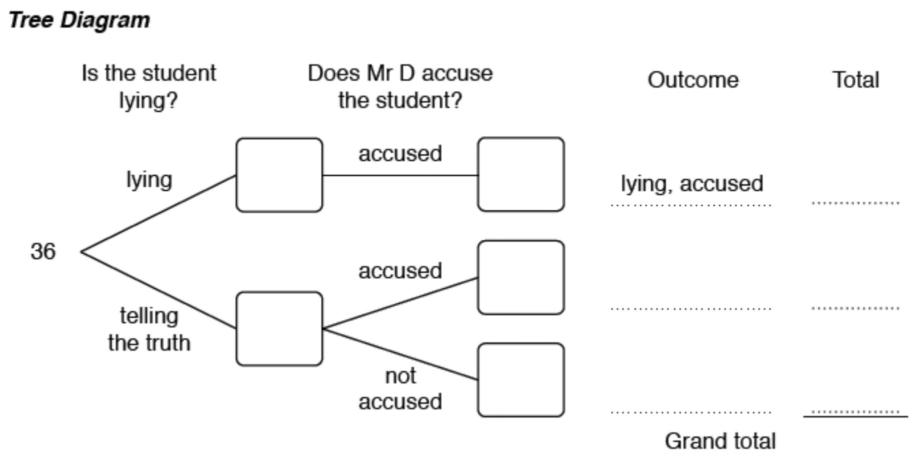


Figure 3. Recording football results

After the initial discussion, groups of students set about collecting data for a season of 36 games, recording their results on the worksheets provided. Many used red and blue pens to record their results, so building a visual impression of how the season was going (Figure 3).

After completing tallies of their results, class discussion included comments on the numbers of wins to each team and the number of draws. Some students expressed surprise at their results, others said their results were as they expected. Groups then entered their results on a tree diagram and a 2-way contingency table, which were provided on the worksheets and were similar to those in Figure 4. Both the tree diagram and the 2-way table were completed using values from the tally – so whole numbers, not probabilities.



Contingency table		Does Mr D accuse the student?		Total
		Accused	Not accused	
Is the student lying?	Lying			
	Telling the truth			
	Total			

Figure 4. Tree diagram and 2-way table from 'The dog ate my homework!' worksheet

‘The dog ate my homework!’ required an ordinary die and multi-link cubes[5] in red, blue, green and yellow for each group of (generally) four students. The scenario is that Mr Detector, the teacher, can always tell when a student is lying about their homework, but there is also a chance that he will accuse a truthful student. The die is thrown once to determine if a student is lying, and a second time if it is necessary to determine if Mr D accuses the student. A 6 on the first throw of the die indicates ‘lying’ and is represented with a red cube, while ‘not lying’ is represented with a blue cube. A 1 on the second throw of the die indicates ‘being accused’ and is represented with a yellow cube, while ‘not being accused’ is represented with a green cube. At the end of the experiment, each group had 36 pairs of cubes – red/yellow (lying and accused), blue/yellow (telling the truth but accused) and blue/green (telling the truth and not accused). The results were recorded in a tally table, then entered into a tree diagram and a 2-way contingency table.

[5] Our use of interlocking cubes derives from research using similar cubes with primary age children in a variety of mathematical areas (Martignon et al., 2007).

Figure 5. 2-way table with pairs of multi-link cubes

At this point, Teacher D put a large sheet of paper out with the 2-way table on it, and invited groups to put their cubes into the correct cells once they had recorded their data for themselves, creating a whole class table (Figure 5).

In all the lessons, once groups completed their tree diagram and 2-way contingency table, the teachers led a whole class discussion about what the students thought their data and the representations of it were telling them. In the teacher's notes, I had emphasised that our approach is to view a tree diagram as a complete set of mutually exclusive narratives for the scenario. Teacher A took this on board, talking his class through the tree diagram as if he were a sports commentator giving the stories for each set of branches. This proved an excellent way of helping the students to understand how the branches relate to each other.

Analysis and interpretation of the classroom observations

For both problems, students had expectations about outcomes right from the start.

Classes investigating the football problem were initially concerned that the dice were unfair:

Teacher C: *Have a look at your dice. What do you notice? What's going on?*[6]

Student 1: It means it's a bit unfair!

Student 2: It might also mean Team Raccoon is better.

Student 3: It's an unfair game, red have more chance of winning than blue!

In all the classes it was initially assumed that as the Raccoons were twice as likely to score, then they were twice as likely to win. The teacher of the lesson I did not observe at School B commented that early on "the misconception of a 1/3 to 2/3 expectation of winning games was strong within the group and written as a conjecture on the board". However, Teacher C, said he thought it was "a super idea. It's very accessible and there's an obvious trap to fall into!", so clearly the initial misconception was not viewed as problematic.

The results did not support the expectation that the Raccoons would win twice as often, and students began to see that the chance of scoring is not the same as the chance of winning the match. Discussion then focused on the draws, both the total number and which team scored

[6] Throughout, transcriptions of comments by teachers are in italics, and comments by students in non-italic font.

first, which second. Most groups of students were surprised that the number of draws was as high as it was. The discussion in School A is typical:

We had more draws than wins, that's a surprise.

About half [of our results] are draws.

Generally students expected that Team Raccoon would score the first goal more often than Team Beaver, and were surprised that several groups' results did not support this conjecture.

Teacher A aggregated the results of nine groups, and the following discussion ensued:

What would you predict if you roll the dice, who would score most?

I'd expect twice as much because it's 4 to 2 on the die.

How many goals would each team score if there were 600 rolls of the die?

400 to 200.

We're pretty close to that. What about results – what do we predict?

Reds will win twice as many games.

No, that doesn't feel right.

Blues didn't even win anywhere near a third of the games, even though they got a third of the goals.

Why doesn't the goal score transfer into games?

Because you have to take account of the draws, scoring a goal doesn't mean a win.

Beavers use up a lot of their goals getting a draw.

This was this class's first introduction to probability other than in descriptive terms, and yet this discussion shows they are capable of a quite sophisticated level of understanding.

The 12-13 year-olds at School B initially conjectured that Raccoons would win twice as many games as Beavers. Then:

> … a couple of the more able students started to use expected results, as they were trying to work through this misconception that had been noted on the board and test it. One student came out and gave a very clear explanation of how he had completed the expected results on his tree, which many other students seemed to be following. To try to convince people of his ideas, we then considered the table … [Email from the teacher]

Even the 10-11 year-olds [School C] were perfectly able to calculate what the expected results ought to be for 36 games.

Classes made predictions before collecting data, some correct, some erroneous, but they were all able to make use of their experimental data to question their predictions, and to explain how the observed results varied from their initial expectations. They were also able to use the tree diagram and the 2-way table to make more informed predictions, which they then used in further analysing their results.

One other aspect worth comment is the way connections were made between the multiple representations. In the lesson I observed in School B, I overheard a group who were ahead of the rest explicitly connecting the numbers in their tally with the numbers on their tree diagram. Teacher C, introducing the tree diagram and 2-way table after groups had collected their data for the lie detector scenario, asked the students which boxes in the tree diagram and 2-way table corresponded to which figures in their tally. In his evaluation, he commented that he wished there had been time for him to talk the students through which figures were common to both representations and which figures were only to be found on one of them, saying that he had never seen before that when both are available, the similarities and differences are much more obvious:

> At what point do you talk about the big things, making explicit the links between the two representations, highlighting the zero, the two dice, that 11 isn't in the tree diagram, but it's the key value?

Evaluating the lessons

It was very encouraging that all the students had enjoyed their lesson and felt they had learnt something. None of them had any difficulty in understanding the scenario, or the instructions for collecting the experimental data. One boy in the 13-14 year-old group at School C said that some people might prefer a theoretical approach over an experimental approach. His teacher commented later that this particular student is dyspraxic, and has particular issues with practical lessons. Other students liked collecting their own data, rather than it being given in a question or by the teacher, claiming that the practical approach helped them to gain insight into the problem. All the students asked were happy with the tally, tree diagram and 2-way table, with classes that had not met tree diagrams before saying they would be fine with them in future. The 10-11 year-old students noticed that the 2-way table was like a Carroll diagram, with which they had long been familiar.

The teachers were also very positive about the lesson they trialled and the overall approach, all commenting on the richness of discussion the problems stimulated. Teacher D particularly liked the kinaesthetic and visual aspects. He also commented on the need for students to use language appropriately in probability, and how a good task brings in this aspect. He said he had had a number of good conversations with individuals during the course of the lesson.

The teacher at School B whose lesson I did not observe wrote in her evaluation:

> The students seemed to gain quite a lot from the lesson through their cooperative work, ownership of the problem (my role was to guide rather than lead) and testing their own conjectures / dispelling their misconceptions. I think this lesson made them realise that investigation is important – they were quick to view the task in an over simplistic way initially which as they explored further they realised was deeper.

All the teachers liked the route from experiment to theoretical analysis, because it avoids the need to explain why the experiment 'didn't work'. Instead students start with a problem and observe what happens, then use the theoretical analysis to help explain the observed results, particularly as larger numbers of results are aggregated.

> The general approach helps with explaining why experimental data is different from the theoretical predictions, which can be awkward. This is the other way round – so there's no need to explain anything. You see a pattern emerging, and the more we do, the more the pattern becomes clear. It's a good process. [Teacher A]

WHAT NEXT?

We are still in the early stages of this project. This pilot of two resources will be developed into a greater range of resources, which will enable students to progress from beginning their study of probability to the point at which they can move onto advanced study. Alongside creating the resources, including computer simulations, we plan to continue to carry out classroom trials and to evaluate the effectiveness of this approach.

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